Unlikely Intersections with Isogeny Orbits

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Prof. Dr. Philipp Habegger
Dr. Gaël Rémond (DR CNRS)

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Prof. Dr. Martin Spiess
Dekan

Contents

CHAPTER 1

Introduction

> Logic, my dear Zoe, merely enables one to be wrong with authority.
>
> The Second Doctor

1.1. Introduction

Why does the equation $2^x \times 2^y = 2$ have infinitely many solutions in integers x and y, while the only solution in integers to $2^x + 2^y = 2$ is $(x, y) = (0, 0)$? This elementary mathematical question can be rephrased as asking about the intersections of two distinct curves inside $\mathbb{G}^2_{m,\bar{\mathbb{Q}}}$ with the finitely generated subgroup $\Gamma = 2^{\mathbb{Z}} \times 2^{\mathbb{Z}}$ of $\mathbb{G}^2_{m,\bar{\mathbb{Q}}}(\bar{\mathbb{Q}})$. One of the two curves is at the same time a translate of an algebraic subgroup, while the other is not.

In another direction, Mordell's conjecture predicted that a smooth projective geometrically irreducible curve C of genus $g \geq 2$, defined over a number field K, has at most finitely many K-rational points. If C has at least one K-rational point, it admits a closed embedding into its Jacobian J so that $C(K) = C \cap J(K)$. Now, the group $J(K)$ is finitely generated by the Mordell-Weil theorem, while C is not a translate of an algebraic subgroup of J since $g \geq 2$.

In a third direction, the Manin-Mumford conjecture asked about the points of $C_{\bar{\mathbb{Q}}}$ that are of finite order in the algebraic group $J_{\bar{\mathbb{Q}}}$. It predicted that their number is finite.

It turns out that all three of these questions are instances of the theorem – proven by McQuillan in [110], following work of Faltings [42], [43], [44], Raynaud [146], Laurent [84], Hindry [72], and Vojta [186] – that for a semiabelian variety G, defined over an algebraically closed field K of characteristic zero, a subgroup Γ of $G(K)$ of finite rank (i.e. satisfying $\dim_{\mathbb{Q}}(\Gamma \otimes_{\mathbb{Z}} \mathbb{Q}) < \infty$), and an irreducible closed subvariety V of G, the intersection $V \cap \Gamma$ is not Zariski dense in V unless V is a translate of an algebraic subgroup of G. Such translates are also called weakly special subvarieties of G. Before it was proven, this statement was called the Mordell-Lang conjecture.

The conjecture of Mordell-Lang admits a natural generalization where one studies the intersection of an irreducible closed subvariety V with translates $\gamma + B$, where γ lies in a subgroup Γ of $G(K)$ of finite rank and B is an algebraic subgroup of G of codimension at least $\dim V + 1$. Such an intersection is deemed unlikely as two generic closed subvarieties do not meet if their codimensions add up to more than the dimension of the ambient space. If the union of these intersections is Zariski dense in V, then the conclusion is no longer that V is a translate itself, but that V is contained in a translate of a proper algebraic subgroup.

If Γ is of rank zero, then we can even conjecture that V is contained in a proper algebraic subgroup. One can show that this conjecture is equivalent to the previous one (see [141], Theorems 5.3 and 5.5). One of the first proofs of a statement in this direction can be found in the pioneering article [22] by Bombieri, Masser and Zannier on powers of the multiplicative group. A conjecture for semiabelian varieties that is on its face stronger than the two aforementioned ones was then formulated by Zilber in [195]. Pink formulated the two aforementioned conjectures together with an analogous conjecture for mixed Shimura varieties in [141]. The conjecture for mixed Shimura varieties actually implies the other two (see Theorem 5.7 in [141]) as well as the André-Oort conjecture. Similar conjectures for $\mathbb{G}_m^n$ were formulated by Bombieri, Masser and Zannier in Section 5 of [23]. The Zilberian formulation in the setting of a mixed Shimura variety or a semiabelian variety is usually referred to as the Zilber-Pink conjecture and forms the centerpiece of the field of unlikely intersections. An overview of this field is given in [193].

Masser and Zannier have studied the Manin-Mumford conjecture in a relative setting where the curve C in its Jacobian J is replaced by a curve $\mathcal{C}$ inside the fibered square of the Legendre family of elliptic curves $\mathcal{E} \to Y(2) = \mathbb{A}^1\backslash\{0,1\}$. They obtained in [103] that there are at most finitely many complex parameters λ for which the points with affine X-coordinates 2 and 3 are both torsion on the elliptic curve given by the affine equation $Y^2 = X(X-1)(X-\lambda)$. See [104], [105], [106], and [107] for later generalizations by the same authors and [175] for Stoll's proof that there are no such λ at all.

One can think of several analogues of the Mordell-Lang conjecture in this relative setting, say of a non-isotrivial abelian scheme $\mathcal{A}$ of relative dimension g over a smooth irreducible curve S with structural morphism π, defined over an algebraically closed field K of characteristic 0. One possibility would be to consider the intersection of all division points of the values taken on $S(K)$ by a finitely generated subgroup of $\mathcal{A}(S)$ with an irreducible closed subvariety $\mathcal{V} \subset \mathcal{A}$. If $K = \bar{\mathbb{Q}}$ and $\mathcal{V}$ is a curve, the results of [10], [11], [12], and [70] can be applied to this problem. Unlike in the case of a constant family of tori (see [3] and [15]), very little is known beyond these results in the abelian case (but see the Appendix of [107]).

Another possibility is to fix an abelian variety A_0 of dimension g, defined over K, and a subgroup Γ of $A_0(K)$ of finite rank and to consider the intersection of $\mathcal{V}$ with the isogeny orbit

$$\mathcal{A}_\Gamma = \{\phi(\gamma); \gamma \in \Gamma, s \in S(K), \text{ and } \phi : A_0 \to \mathcal{A}_s \text{ is an isogeny}\}.$$

Here and in the following, $\mathcal{A}_s$ denotes the fiber of $\mathcal{A}$ over s. One can also allow translates of algebraic subgroups and consider (for fixed $k \in \mathbb{N} \cup \{0\}$) the intersection with the $(g-k)$-enlarged isogeny orbit

$$\begin{aligned}\mathcal{A}_\Gamma^{[k]} = \{\phi(\gamma + b); \gamma \in \Gamma, s \in S(K), b \in B(K),\ & B \subset A_0 \text{ is an abelian subvariety} \\ & \text{of codimension} \geq k, \text{ and } \phi : A_0 \to \mathcal{A}_s \text{ is an isogeny}\}.\end{aligned}$$

If the intersection is Zariski dense in $\mathcal{V}$, one again wants to conclude that $\mathcal{V}$ is "weakly special" in a suitable sense or contained in a "weakly special" subvariety of $\mathcal{A}$ of codimension $\geq k$ respectively. This fits into the general philosophy of the Zilber-Pink conjecture.

For $k = \dim A_0$, we call the corresponding statement the modified André-Pink-Zannier conjecture over a curve. It is a generalization of Conjecture 1.2 in [51], called the André-Pink-Zannier conjecture, in the case where the base variety is a curve. The modified André-Pink-Zannier conjecture over a curve follows naturally from Pink's Conjecture 1.6 in [140] on the intersection of irreducible closed subvarieties of mixed Shimura varieties with generalized Hecke orbits (see Sections 4 and 8 of [51]), which in turn follows from Pink's Conjecture 1.1 for mixed Shimura varieties in [141]. If Γ is of rank zero, the (modified) André-Pink-Zannier conjecture (over a curve) is contained in a conjecture of Zannier (Conjecture 1.4 in [51]). We note here that the André-Pink-Zannier conjecture concerns a subvariety of a certain universal family of abelian varieties and that its conclusion also contains a (strong) condition that the projection of this subvariety to the corresponding moduli space has to satisfy. However, if the subvariety projects to a curve in the moduli space, this part of the conjecture is known thanks to Orr [124]. See Section 3.1 for a precise statement and more thorough discussion of the modified André-Pink-Zannier conjecture.

In Chapter 3 (corresponding to [38]), we prove this conjecture for a curve $\mathcal{C}$ inside a non-isotrivial abelian scheme $\mathcal{A}$ and even characterize the curves that dominate the base and potentially intersect $\mathcal{A}_\Gamma^{[k]}$ infinitely often for some given $k \in \mathbb{N} \cup \{0\}$, under the condition that $K = \bar{\mathbb{Q}}$. In this setting, let ξ denote the generic point of S and let $\left(\mathcal{A}_\xi^{\overline{\bar{\mathbb{Q}}(S)}/\bar{\mathbb{Q}}}, \mathrm{Tr}\right)$ denote the $\overline{\bar{\mathbb{Q}}(S)}/\bar{\mathbb{Q}}$-trace of $\mathcal{A}_\xi$ as defined in Chapter VIII, §3 of [81] for a fixed algebraic closure $\overline{\bar{\mathbb{Q}}(S)}$ of $\bar{\mathbb{Q}}(S)$. Here and throughout the thesis, we freely identify algebraic varieties with their base change to a fixed algebraic closure of their field of definition as well as with the closed points of said base change. We call $\mathcal{A} \to S$ isotrivial if Tr is surjective. We obtain the following theorems:

THEOREM 1.1.1. *(= Theorem 3.1.2) Suppose that* $K = \bar{\mathbb{Q}}$ *and* $\mathcal{A} \to S$ *is not isotrivial. If* $\mathcal{A}_\Gamma^{[k]} \cap \mathcal{C}$ *is infinite and* $\pi(\mathcal{C}) = S$*, then* $\mathcal{C}$ *is contained in an irreducible closed subvariety* $\mathcal{W}$ *of* $\mathcal{A}$ *of codimension* $\geq k$ *with the following property: Over* $\overline{\bar{\mathbb{Q}}(S)}$*, every irreducible component of* $\mathcal{W}_\xi$ *is a translate of an abelian subvariety of* $\mathcal{A}_\xi$ *by a point in* $(\mathcal{A}_\xi)_{\mathrm{tors}} + \mathrm{Tr}\left(\mathcal{A}_\xi^{\overline{\bar{\mathbb{Q}}(S)}/\bar{\mathbb{Q}}}(\bar{\mathbb{Q}})\right)$.

THEOREM 1.1.2. *(= Theorem 3.1.3) Suppose that* $K = \bar{\mathbb{Q}}$ *and* $\mathcal{A} \to S$ *is not isotrivial. If* $\mathcal{A}_\Gamma \cap \mathcal{C}$ *is infinite, then one of the following two conditions is satisfied:*

(i) The curve $\mathcal{C}$ *is a translate of an abelian subvariety of* $\mathcal{A}_s$ *by a point of* $\mathcal{A}_\Gamma \cap \mathcal{A}_s$ *for some* $s \in S(\bar{\mathbb{Q}})$*.*

(ii) The zero-dimensional variety $\mathcal{C}_\xi$ *is contained in* $(\mathcal{A}_\xi)_{\mathrm{tors}} + \mathrm{Tr}\left(\mathcal{A}_\xi^{\overline{\bar{\mathbb{Q}}(S)}/\bar{\mathbb{Q}}}(\bar{\mathbb{Q}})\right)$*.*

Previous related results have been obtained by Lin and Wang in [88], by Habegger in [68], by Pila in [133], and by Gao in [51]; in the last two articles, there is no restriction on the field K. As a corollary of Theorem 1.1.2, we prove a generalization of a conjecture by Buium and Poonen (Conjecture 1.7 in [27]); the conjecture has been proven independently by Baldi in [8].

Our proof of Theorem 1.1.1 follows the Pila-Zannier strategy that originates in the new proof of the Manin-Mumford conjecture by Pila and Zannier in [138]. We briefly outline this strategy: To each point p that arises in an unlikely intersection, we can associate a point q on a certain subset of some $\mathbb{R}^n$ that is definable in the o-minimal structure $\mathbb{R}_{\mathrm{an,exp}}$. In order to define this set, we use a certain transcendental uniformization map. Certain coordinates of q are integers; their maximum in absolute value is often called the complexity of p. By taking Galois conjugates over a fixed number field over which all the data is defined, a point p of large degree gives rise to many such points. If we can bound the complexity of p polynomially in its degree, i.e. establish so-called "Galois bounds", then a powerful point-counting result from o-minimality proven by Pila and Wilkie in [137] (or more precisely a later variant due to Habegger and Pila in [70]) allows us to deduce that the given definable set must contain a definable curve on which certain coordinates satisfy algebraic relations while its projection on another coordinate subspace is positive-dimensional. We then need a functional transcendence result to conclude that the curve $\mathcal{C}$ must satisfy the conclusion of the theorem.

The necessary functional transcendence result here has been proven in [52] by Gao. For the Galois bounds, we need the degree bounds for isogenies established by Masser and Wüstholz in [100] together with Faltings' bound on the difference between the Faltings heights of two isogenous abelian varieties in [42] and the bounds for the difference between the "Theta height" and the Faltings height in [126]; Faltings' bound was used already by Masser and Wüstholz to establish their degree bounds. We cleverly choose one specific isogeny of minimal degree, following Orr [124]. The Galois bounds are then obtained by combining results of Habegger and Pila in [70], Rémond in [149] and [156], Masser in [95], and some elementary diophantine approximation. The definability of the transcendental uniformization map follows from results of Peterzil and Starchenko in [128].

If we restrict ourselves to certain fibered products of elliptic schemes (and $K = \bar{\mathbb{Q}}$), we can prove the modified André-Pink-Zannier conjecture over a curve in full generality. In order to obtain the height bounds that are needed for the Galois bounds in the Pila-Zannier strategy, we apply a generalized Vojta-Rémond inequality. In Chapter 4 (corresponding to [39]), we prove a slightly more general version of the following theorem:

THEOREM 1.1.3. *(= Theorem 4.1.1) Suppose that $K = \bar{\mathbb{Q}}$, that $\mathcal{A} \to S$ is not isotrivial, and that over $\overline{\bar{\mathbb{Q}}(S)}$, $\mathcal{A}_\xi$ is isogenous to a power of an elliptic curve. Suppose further that A_0 is isogenous to E_0^g, where E_0 is an elliptic curve with* $\mathrm{End}(E_0) = \mathbb{Z}$.

Let $\mathcal{V} \subset \mathcal{A}$ be an irreducible closed subvariety. If $\mathcal{A}_\Gamma \cap \mathcal{V}$ is Zariski dense in $\mathcal{V}$, then one of the following two conditions is satisfied:

- *(i) The variety $\mathcal{V}$ is a translate of an abelian subvariety of $\mathcal{A}_s$ by a point of $\mathcal{A}_\Gamma \cap \mathcal{A}_s$ for some $s \in S(\bar{\mathbb{Q}})$.*
- *(ii) Over $\overline{\bar{\mathbb{Q}}(S)}$, the variety $\mathcal{V}_\xi$ is a union of translates of abelian subvarieties of $\mathcal{A}_\xi$ by points in $(\mathcal{A}_\xi)_{\mathrm{tors}}$.*

The restrictions on the abelian scheme $\mathcal{A}$ and on A_0 come from the fact that we need a lower bound for a certain intersection number in order to apply the generalized Vojta-Rémond inequality. It is unclear how to obtain such a bound without the restriction. Compared to the curve case, most of the work is spent on obtaining the substantially more difficult height bounds through use of the generalized Vojta-Rémond inequality. If $\mathcal{V}$ contains a Zariski dense set of translates of positive-dimensional abelian subvarieties of fibers of $\mathcal{A} \to S$, then a direct application of the generalized Vojta-Rémond inequality might not lead to success. However, this case can be avoided by "dividing out" the stabilizer of the generic fiber of $\mathcal{V}$ (after maybe making a finite surjective base change $S' \to S$). Once the height bound is established, the proof runs along the same lines as in the curve case. Since we work in a product of elliptic schemes, we can use a functional transcendence result of Pila in [133].

If Γ is of rank zero, then the analogue of Theorem 1.1.3 has been proven for general A_0 and (non-isotrivial) $\mathcal{A}$ and $K = \mathbb{C}$ by Gao in [51]. Previously, results had been obtained in the rank-zero case under restrictions on $\mathcal{A} \to S$ by Habegger in [68] and by Pila in [133]; in the latter article, the base variety S is even allowed to have arbitrary dimension as long as $\mathcal{A} \to S$ is of a certain special form. Habegger does not use the Pila-Zannier strategy in his article, but uses a height inequality instead, which he and Gao later generalized in [55] and which has interesting applications beyond the problem discussed here. If one uses the Pila-Zannier strategy however, then none of the above-mentioned difficulties in establishing the height bounds arise in this case since the canonical height of a torsion point is zero.

In Section 5.1, we sketch how to remove the condition in Theorem 1.1.2 that everything is defined over the field of algebraic numbers. This is achieved through use of the Moriwaki height and specialization arguments. All the necessary arguments are contained essentially already in Gao's article [51].

In certain cases, Theorem 1.1.2 can be made effective by 2-adic considerations in the style of Stoll and Mavraki (see [175] and [109]), and then it shows that the corresponding intersection is not only unlikely, but actually impossible. We do this in Section 5.2, using a result of Stoll and Mavraki.

In Section 5.3, we apply Theorem 1.1.2 to classify semiabelian schemes over a curve with infinitely many isogenous fibers; the analogous question for abelian schemes, defined over $\mathbb{C}$, was answered by Orr in [124]. If everything is defined over $\bar{\mathbb{Q}}$, we show that the geometric generic fiber of a semiabelian scheme of finite type over a curve with infinitely many pairwise isogenous fibers is either an extension of the base change of an abelian variety defined over $\bar{\mathbb{Q}}$ by a torus or is isogenous to the product of the base change of a semiabelian variety defined over $\bar{\mathbb{Q}}$ with an abelian variety.

In Section 5.4, we apply the well-known strategy for proving the Manin-Mumford conjecture by intersecting with Galois conjugates to the André-Pink-Zannier conjecture in the case that Γ has rank zero. Our main tool is Serre's theorem on the intersection of the image of the adelic Galois representation with the group $\hat{\mathbb{Z}}^*$ of homotheties (see No. 136 in [167]). We follow the strategy of Hindry in [72], which goes back to Lang, Serre, and Tate [82], and apply some of his results. This approach has the advantage of being able to yield effective results in certain cases thanks to the work of Lombardo ([89], [90], [91], [92], [93]). It also allows the base variety S to be of arbitrary dimension as long as the abelian scheme has maximal variation. We obtain the following theorem:

THEOREM 1.1.4. *(= Theorem 5.4.1) Let S be an irreducible affine variety, defined over $\bar{\mathbb{Q}}$, and let ξ denote the generic point of S. Let $\pi : \mathcal{A} \to S$ be a principally polarized abelian scheme of relative dimension g over S, also defined over $\bar{\mathbb{Q}}$. Suppose that the natural morphism $\rho : S \to A_g$ to the coarse moduli space A_g of principally polarized abelian varieties of dimension g is quasi-finite.*

Let $\mathcal{V} \subset \mathcal{A}$ be an irreducible closed subvariety such that $\pi(\mathcal{V}) = S$. Fix an abelian variety A_0, defined over $\bar{\mathbb{Q}}$. Suppose that the set of $x \in \mathcal{V}(\bar{\mathbb{Q}})$ such that x is a torsion point of the fiber $\mathcal{A}_{\pi(x)}$ and such that $\mathcal{A}_{\pi(x)}$ is isogenous to A_0 is Zariski dense in $\mathcal{V}$. Then $\mathcal{V}_\xi$ is equal to a union of translates of abelian subvarieties of $\mathcal{A}_\xi$ by torsion points of $\mathcal{A}_\xi$ (over $\overline{\bar{\mathbb{Q}}(S)}$).

We thus prove one half of Conjecture 1.4 in [51], due to Zannier, in the case where everything is defined over $\bar{\mathbb{Q}}$. In order to establish the full conjecture over $\bar{\mathbb{Q}}$, one would need to prove that the Zariski closure of $\rho(S) \subset A_g$ is a totally geodesic (or, equivalently, weakly special) subvariety of A_g. Theorem 1.1.4 also holds if ρ is just assumed to be generically finite and we drop the assumption that S is affine since we can then replace S by an open affine Zariski dense subset restricted to which ρ is quasi-finite.

If $\mathcal{A}$ is contained in a product of elliptic modular surfaces (as defined in [133]), then the analogue of Zannier's conjecture has been proven by Pila in [133]. If S is a curve or A_0 has CM, then the conjecture has been proven by Gao in [51]. Previously, Habegger proved the analogue of the conjecture in [68] if $\mathcal{A}$ is the fibered power of an elliptic scheme over a smooth quasi-projective base curve, defined over $\bar{\mathbb{Q}}$, and A_0 is defined over $\bar{\mathbb{Q}}$. In the works of Gao and Pila, the field of definition is allowed to be $\mathbb{C}$ instead of $\bar{\mathbb{Q}}$. It would be interesting to know whether Theorem 1.1.4 could also be extended by specialization arguments to the case where everything is defined over $\mathbb{C}$.

In Section 5.5, we explore sufficient conditions for obtaining the lower bound for a certain intersection number that we need to apply the generalized Vojta-Rémond inequality in Chapter 4.

In Chapter 6, which is joint work with Fabrizio Barroero (corresponding to [13]), we show how to reduce the Zilber-Pink conjecture for abelian varieties over an arbitrary algebraically closed field K of characteristic zero to the case $K = \bar{\mathbb{Q}}$. We prove the following theorem:

THEOREM 1.1.5. *(= Theorem 6.1.5) Let K be an algebraically closed field of characteristic 0, let m be a non-negative integer and A an abelian variety defined over K with $K/\bar{\mathbb{Q}}$-trace (T, Tr). If for some non-negative integer d every irreducible closed subvariety of T of dimension at most m contains at most finitely many optimal subvarieties of defect at most d, then every irreducible closed subvariety of A of dimension at most m contains at most finitely many optimal subvarieties of defect at most d as well.*

See Chapter 6 for the definition of an optimal subvariety and its defect. Combining this theorem with Theorem 1.1 of Habegger-Pila in [70], we obtain a proof of the Zilber-Pink conjecture for irreducible curves in abelian varieties over an arbitrary algebraically closed field of characteristic zero. So far, the conjecture was only known if both the curve and the abelian variety are defined over the algebraic numbers by said theorem of Habegger-Pila. We also obtain a proof of the conjecture if the dimension of the $K/\bar{\mathbb{Q}}$-trace of the abelian variety is at most 4, so for example if the abelian variety is an arbitrary power of an elliptic curve with transcendental j-invariant. Furthermore, we show in Theorem 6.1.9 that the apparently stronger Zilberian formulation of the Zilber-Pink conjecture for abelian varieties is in fact implied by Pink's formulation.

The analogue of Theorem 1.1.5 for powers of the multiplicative group has been proven by Bombieri, Masser and Zannier in [24]. Our proof at some points resembles theirs, albeit formulated rather differently. In particular, we also make crucial use of what is sometimes called a "Structure Theorem". For powers of the multiplicative group, this was established by Poizat in [143] and independently by Bombieri, Masser and Zannier in [23]. We use the corresponding results for abelian varieties, due to Rémond in [157], and for connected mixed Shimura varieties of Kuga type, due to Gao in [54].

In Appendix A (corresponding to [37]), we prove the generalized Vojta-Rémond inequality that is applied in Chapter 4. This appendix draws heavily on a generalization of Rémond's generalized Vojta inequality [154] by Ange in [5]. However, Ange's generalization was not quite sufficient for our purposes and we had to generalize it somewhat further. Compared to Rémond's work, the main new feature is that the inequality can be applied to a product of distinct irreducible projective varieties instead of a power of one fixed irreducible projective variety. This is indispensable for our application in Chapter 4, where we consider a product of irreducible components of fibers $\mathcal{V}_s$ ($s \in S(\bar{\mathbb{Q}})$). Of course, all these results ultimately owe their existence to Vojta's original work [186].

In many proofs in this thesis, the logarithmic absolute Weil height $h : \bar{\mathbb{Q}} \to [0, \infty)$, which provides a measure for the complexity of an algebraic number, is a useful technical tool. In Appendix B, we study it as an object of interest in its own right and ask ourselves how often it assumes certain values. Schanuel counted algebraic numbers of bounded height in a fixed number field in [162]. Masser and Vaaler then counted algebraic numbers of fixed degree and bounded height in [98] (over $\mathbb{Q}$) and [97] (over any number field). We count instead algebraic numbers of fixed degree and fixed height. Here, the growth behaviour is necessarily less uniform, but one can still make rough qualitative statements about it.

To illustrate the flavor of our results: If ϕ denotes Euler's phi function, then $\lim_{n\to\infty} \frac{\phi(n)}{n}$ does not exist, but $\lim_{n\to\infty} \frac{\log \phi(n)}{\log n} = 1$. This is related to counting algebraic numbers of degree 1 and fixed height. For degree $d > 1$, we have found it necessary to take into account also the number k of conjugates of the algebraic number that lie inside the open unit disk. However, even for fixed k and d and after one has taken the logarithm, it can happen that the limit fails to exist. Our methods are mostly elementary; to construct many algebraic numbers of given height and degree, we use point-counting results for lattices due to Barroero and Widmer in [14] (generalizing a classical result of Davenport in [35]) and Technau and Widmer in [177].

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CHAPTER 2

Preliminaries and notation

But yet I'll make assurance double sure.

W. Shakespeare, *Macbeth*

The following conventions will be followed throughout this thesis:

2.1. Generalities

The natural numbers are the set $\mathbb{N} = \{1, 2, 3, \ldots\}$.

We fix once and for all a square root of -1 inside $\mathbb{C}$ that we denote by $\sqrt{-1}$; this yields maps $\mathrm{Re} : \mathbb{C} \to \mathbb{R}$ and $\mathrm{Im} : \mathbb{C} \to \mathbb{R}$ in the usual way. For an integral domain R, we denote the ring of $m \times n$-matrices with entries in R by $\mathrm{M}_{m\times n}(R)$. We write $\mathrm{M}_n(R)$ for $\mathrm{M}_{n\times n}(R)$. The group of units of a ring R is denoted by R^*.

The complex conjugate of a matrix A with complex entries is denoted by $\overline{A}$ and the transpose by A^t. The n-dimensional identity matrix is denoted by I_n. The row-sum norm of a matrix $A \in \mathrm{M}_{m\times n}(\mathbb{C})$ is denoted by $\|A\|$.

For a vector $v = (v_1, \ldots, v_n)^t \in \mathbb{C}^n$, we write $\|v\|$ for $\max_{j=1,\ldots,n} |v_j|$. Note that

$$\|A\| = \max_{v\neq 0} \frac{\|Av\|}{\|v\|}$$

for all $A \in \mathrm{M}_{m\times n}(\mathbb{C})$. Vectors are always column vectors. By applying Re and Im to each entry, we obtain maps from $\mathrm{M}_n(\mathbb{C})$ to $\mathrm{M}_n(\mathbb{R})$ that by abuse of language are also called Re and Im.

The upper half plane $\mathbb{H}$ is the set of $\tau \in \mathbb{C}$ with $\mathrm{Im}\,\tau > 0$.

2.2. Algebraic geometry

A variety over a field K or K-variety is a reduced and separated $\mathrm{Spec}\,K$-scheme of finite type. Morphisms between K-varieties are morphisms of K-schemes. The field K will always be of characteristic 0 so that reducedness is equivalent to geometric reducedness. We will always fix an algebraic closure $\bar{K}$ of K.

If F is any field extension of K, we denote the set of F-points of a K-variety V by $V(F)$.

Subvarieties will always be closed.

We say that a K-variety is smooth or non-singular if it is smooth over $\mathrm{Spec}\,K$.

If V and W are two K-varieties, we write $V \times W$ for $V \times_K W$.

We say that a variety is defined over some subfield $L \subset K$ if it is obtained as the base change of an L-variety to K. A morphism is said to be defined over L if the source and the target are defined over L and the morphism is the base change of a morphism of L-varieties. A subvariety of a variety is said to be defined over L

if the subvariety and its closed embedding are both defined over L. An open subset of a variety V is said to be defined over L if the corresponding variety and its open immersion into V are both defined over L.

If $\sigma \in \mathrm{Gal}(\bar{K}/K)$ and V is a $\bar{K}$-variety, we define $\sigma(V)$ as the $\bar{K}$-variety obtained by composing the structural morphism $V \to \operatorname{Spec} \bar{K}$ with the automorphism of $\operatorname{Spec} \bar{K}$ that is induced by σ^{-1}. If W is a $\bar{K}$-variety, then a morphism $\phi : W \to V$ naturally induces a morphism $\sigma(\phi) : \sigma(W) \to \sigma(V)$. If V is defined over K, then $\sigma(V)$ is naturally isomorphic to V.

A morphism $\phi : W \to V$ of algebraic varieties over K is called projective (resp. quasi-projective) if it factorizes as the composition of a closed embedding (resp. an immersion) $W \hookrightarrow \mathbb{P}^N_K \times V$ with the projection $\mathbb{P}^N_K \times V \to V$ (for some $N \in \mathbb{N}$). This definition coincides with the one of Hartshorne in [71], but is slightly different from the one in [62] or [58]. If S is affine, then all the definitions are equivalent. Likewise, a line bundle on W is called very ample relative to V if it is isomorphic to the pull-back of $\mathcal{O}(1)$ under a V-immersion $W \hookrightarrow \mathbb{P}^N_K \times V$ for some $N \in \mathbb{N}$ (this definition agrees with the one in [71]). A line bundle on W is called very ample if it is very ample relative to $\operatorname{Spec} K$.

2.3. Heights

We use the logarithmic absolute Weil height h on projective space $\mathbb{P}^n(\bar{\mathbb{Q}})$ as defined in Definition 1.5.4 in [20] by use of the maximum norm at the infinite places. We also use its exponential counterpart $H = \exp \circ h$.

By restricting to $\bar{\mathbb{Q}} = \mathbb{A}^1(\bar{\mathbb{Q}}) \subset \mathbb{P}^1(\bar{\mathbb{Q}})$, we obtain the usual absolute Weil height of an algebraic number. The height of a finite subset of $\bar{\mathbb{Q}}$ is defined by considering it as a point in an appropriate projective space.

The height of the algebraic number $\alpha \in \bar{\mathbb{Q}}$ is therefore equal to the height of the subset $\{1, \alpha\}$, but in general not equal to the height of the subset $\{\alpha\}$. Making matters worse, we will sometimes also associate a height to the subvariety $\{\alpha\}$ of $\mathbb{P}^1_{\bar{\mathbb{Q}}}$, which in general is equal to neither of the two aforementioned heights. Still, there should never be any confusion about which height we are using.

We always use upper-case letters for exponential heights and lower-case letters for logarithmic heights.

If $\alpha = \frac{a}{b}$ is a rational number with $a \in \mathbb{Z}$, $b \in \mathbb{N}$ and $\gcd(a, b) = 1$, then we have $H(\alpha) = \max\{|a|, |b|\}$. For a matrix $A = (a_{ij})_{1 \leq i,j \leq n} \in \mathrm{M}_n(\mathbb{Q})$, we define its height $H(A) = \max_{1 \leq i,j \leq n} H(a_{ij})$.

If V is a projective variety over $\bar{\mathbb{Q}}$ and L a very ample line bundle on V, then any closed embedding of V into projective space associated to L yields an associated height $h_{V,L} : V(\bar{\mathbb{Q}}) \to [0, \infty)$, induced by the logarithmic height on projective space. It will always be clear from the context which embedding we are choosing.

2.4. Abelian schemes and varieties

An abelian scheme $\mathcal{A}$ over a K-variety S is a smooth and proper group scheme over S with geometrically connected fibers. As the morphism $\mathcal{A} \to S$ is smooth and S is reduced, the scheme $\mathcal{A}$ is reduced as well (see [65], Proposition 11.3.13(ii)) and is therefore also a K-variety.

Typically, the structural morphism $\mathcal{A} \to S$ will be denoted by π and the zero section $S \to \mathcal{A}$ by ϵ. The section ϵ is a closed embedding since π is separated.

A subgroup scheme (resp. an open subgroup scheme, resp. a closed subgroup scheme) $\mathcal{B}$ of $\mathcal{A}$ is a subscheme (resp. an open subscheme, resp. a closed subscheme) of $\mathcal{A}$ such that the zero section $S \to \mathcal{A}$ factors through $\mathcal{B}$ and the morphisms $\mathcal{B} \times_S \mathcal{B} \to \mathcal{A}$ and $\mathcal{B} \to \mathcal{A}$ that are induced by the addition and inversion morphism of $\mathcal{A}$ respectively factor through $\mathcal{B}$.

We now suppose that S is irreducible. An irreducible subgroup scheme $\mathcal{B}$ of $\mathcal{A}$ is called an abelian subscheme if $\mathcal{B} \to S$ is flat, proper, and dominant. Equivalently, an abelian subscheme is an irreducible closed subgroup scheme that is flat over S. An abelian subscheme $\mathcal{B}$ is itself an abelian scheme over S: For each natural number N, the multiplication-by-N morphism from $\mathcal{B}$ to $\mathcal{B}$ is dominant and proper, hence surjective. It follows that the geometric fibers of $\mathcal{B}$ must be connected as desired.

An abelian variety over a field K is an abelian scheme over the K-variety $\operatorname{Spec} K$. A line bundle on an abelian variety is called symmetric if it is isomorphic to its pull-back under the inversion morphism.

A morphism between two abelian varieties is called an isogeny if it is a finite surjective homomorphism of algebraic groups. Two abelian varieties are called isogenous if there exists an isogeny between them. If A is an abelian variety, we denote the set of its torsion points over an algebraic closure of its field of definition by A_{tors} and its zero element by 0_A or just 0 if there is no potential confusion.

If A is an abelian variety and $V \subset A$ a subvariety, we denote its stabilizer by

$$\operatorname{Stab}(V, A) := \{a \in A; a + V \subset V\},$$

where we identify the varieties with their sets of closed points over an algebraic closure of their field of definition. As $\operatorname{Stab}(V, A) = \bigcap_{x \in V} (-x + V)$, it is Zariski closed. By considering each irreducible component of V separately, we find that $a \in \operatorname{Stab}(V, A)$ actually implies that $a + V = V$. Hence, the stabilizer is also closed under addition and inversion, and therefore is an algebraic subgroup of A.

If A is an abelian variety, we denote its dual abelian variety by $\hat{A}$. If $\phi : A \to B$ is a homomorphism of algebraic groups, the dual homomorphism is denoted by $\hat{\phi} : \hat{B} \to \hat{A}$.

2.5. Mixed Shimura varieties and fine moduli spaces

The only connected mixed Shimura varieties that appear in this dissertation are the universal families of abelian varieties over the fine moduli spaces classifying principally polarized abelian varieties of given dimension with given symplectic or so-called orthogonal level structure. In this section, we sketch how they are constructed, following Pink [139] and [140]. For the definition of connected mixed Shimura data and varieties as well as of morphisms between connected mixed Shimura data and Shimura morphisms between connected mixed Shimura varieties, see Definitions 2.1, 2.4, 2.5 and 2.7 in [140].

Let $P = V_{2g} \rtimes \mathrm{GSp}_{2g}$, where $V_{2g} = \mathbb{Q}^{2g}$,

$$\mathrm{GSp}_{2g} = \{M \in \mathrm{GL}(V_{2g}); \exists \nu(M) \in \mathbb{G}_m : \Psi(Mv, Mw) = \nu(M)\Psi(v, w) \forall v, w \in V_{2g}\},$$

and Ψ is the non-degenerate alternating form on V_{2g} that is given by the matrix $\begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}$. We will write elements of $P(\mathbb{C})$ as (M,v) with $M \in \mathrm{GSp}_{2g}(\mathbb{C})$ and $v \in V_{2g}(\mathbb{C})$.

Let $\mathrm{GSp}_{2g}(\mathbb{R})^+$ denote the connected component of I_{2g} in $\mathrm{GSp}_{2g}(\mathbb{R})$ and let $\mathbb{S}$ denote the restriction of scalars of $\mathbb{G}_{m,\mathbb{C}}$ from $\mathbb{C}$ to $\mathbb{R}$ so that $\mathbb{S}(\mathbb{R})$ is in natural bijection with $\mathbb{C}\backslash\{0\}$. Let ρ denote the base change to $\mathbb{C}$ of the homomorphism $\mathbb{S} \to P_{\mathbb{R}}$ that sends $r + s\sqrt{-1} \in \mathbb{S}(\mathbb{R})$ to $\left(rI_{2g} + s\begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}, 0\right) \in P(\mathbb{R})$, where $(r,s) \in \mathbb{R}^2\backslash\{(0,0)\}$. Let $\mathcal{X}^+$ be the $V_{2g}(\mathbb{R}) \rtimes \mathrm{GSp}_{2g}(\mathbb{R})^+$-conjugacy class of the homomorphism ρ, where the action by conjugation is from the left. Then $(P, \mathcal{X}^+)$ is a connected mixed Shimura datum (see [140], Example 2.12).

There is a canonical structure of a holomorphic manifold on $\mathcal{X}^+$ (see [139], Propositions 1.7 and 1.16). To compute it explicitly, we choose the faithful representation $\phi : P \to \mathrm{GL}_{2g+1}$ defined by $(M,v) \mapsto \begin{pmatrix} M & v \\ 0 & 1 \end{pmatrix}$. For each $\rho' \in \mathcal{X}^+$, we get a homomorphism $\phi_{\mathbb{C}} \circ \rho' : \mathbb{S}_{\mathbb{C}} \to \mathrm{GL}_{2g+1,\mathbb{C}}$.

This map induces a decomposition $\mathbb{C}^{2g+1} = V^{0,-1} \oplus V^{-1,0} \oplus V^{0,0}$, where $V^{p,q}$ is the vector subspace of all $w \in \mathbb{C}^{2g+1}$ such that $(\phi_{\mathbb{C}} \circ \rho')(z)w = z^{-p}\overline{z}^{-q}w$ for $z \in \mathbb{S}(\mathbb{R}) = \mathbb{C}\backslash\{0\}$. Suppose that

$$(\phi_{\mathbb{C}} \circ \rho')\left(r + s\sqrt{-1}\right) = \begin{pmatrix} rI_{2g} + sM & ((1-r)I_{2g} - sM)v \\ 0 & 1 \end{pmatrix}$$

for $r + s\sqrt{-1} \in \mathbb{S}(\mathbb{R})$ ($(r,s) \in \mathbb{R}^2\backslash\{(0,0)\}$), i.e. ρ' is obtained from ρ by conjugating with $(I_{2g}, v)(U, 0)$, where $U \in \mathrm{GSp}_{2g}(\mathbb{R})^+$ is chosen such that $U\begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}U^{-1} = M$.

We compute that

$$V^{0,0} = \left\{ \begin{pmatrix} av \\ a \end{pmatrix}; a \in \mathbb{C} \right\}, \quad V^{0,-1} = \left\{ \begin{pmatrix} \tau \\ I_g \\ 0 \end{pmatrix} w; w \in \mathbb{C}^g \right\},$$

$$V^{-1,0} = \left\{ \begin{pmatrix} \overline{\tau} \\ I_g \\ 0 \end{pmatrix} w; w \in \mathbb{C}^g \right\}.$$

Here, $\tau \in M_g(\mathbb{C})$ is a period matrix associated with M. It is uniquely characterized by $(-\sqrt{-1})\begin{pmatrix} \tau & I_g \end{pmatrix} = \begin{pmatrix} \tau & I_g \end{pmatrix} M^t$. Every τ thus obtained lies in $\mathbb{H}_g = \{\tau \in M_g(\mathbb{C}); \tau^t = \tau \text{ and } \operatorname{Im}\tau \text{ is positive definite}\}$ and vice versa, every element of $\mathbb{H}_g$ can be obtained in this manner.

We set $F^0 = V^{0,0} \oplus V^{0,-1}$; it is a vector subspace of $\mathbb{C}^{2g+1}$ of dimension $g+1$. If $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ with $v_1, v_2 \in \mathbb{R}^g$, then

$$F^0 = \left\{ \begin{pmatrix} \tau & v_1 - \tau v_2 \\ I_g & 0 \\ 0 & 1 \end{pmatrix} w; w \in \mathbb{C}^{g+1} \right\}. \tag{2.5.1}$$

We get a map from $\mathcal{X}^+$ into the Grassmannian that parametrizes $(g+1)$-dimensional vector subspaces of $\mathbb{C}^{2g+1}$ by sending ρ' to F^0. The holomorphic structure on $\mathcal{X}^+$ is inherited from its image in this Grassmannian (see [139], Proposition 1.7). By (2.5.1), the image is contained in an affine piece of the Grassmannian and can be identified (biholomorphically) with $\mathbb{H}_g \times \mathbb{C}^g$.

We can now explain the choice of the negative sign in the definition of τ: It implies that $\sqrt{-1}\begin{pmatrix} I_g & -\tau \end{pmatrix} = \begin{pmatrix} I_g & -\tau \end{pmatrix} M$. Therefore, M describes multiplication by $\sqrt{-1}$ on the fiber $\{\tau\} \times \mathbb{C}^g \simeq \mathbb{R}^{2g}$ with respect to the standard basis on $\mathbb{R}^{2g}$, where the isomorphism is given by sending $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in \mathbb{R}^{2g}$ to $v_1 - \tau v_2 \in \mathbb{C}^g$.

Next, we compute the action of $V_{2g}(\mathbb{R}) \rtimes \mathrm{GSp}_{2g}(\mathbb{R})^+$ on $\mathbb{H}_g \times \mathbb{C}^g$ that is induced by its action on $\mathcal{X}^+$ by conjugation. We obtain that $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{GSp}_{2g}(\mathbb{R})^+$ acts by sending (τ, z) to $(M[\tau], \nu(M)(C\tau + D)^{-t}z)$, where $M[\tau] = (A\tau + B)(C\tau + D)^{-1}$ and $\nu(M) \in \mathbb{R}_{>0}$ is the multiplier associated to $M \in \mathrm{GSp}_{2g}(\mathbb{R})^+$. On the other hand, $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in V_{2g}(\mathbb{R})$ acts by sending (τ, z) to $(\tau, z + v_1 - \tau v_2)$.

We recall the following definitions from Section 0.5 of [139]: Let $n \in \mathbb{N}$. An element of $\mathrm{GL}_n(\mathbb{Q})$ is called neat if the subgroup of $\bar{\mathbb{Q}}^\times$ that is generated by its eigenvalues is torsion-free. If G is a linear algebraic group over $\mathbb{Q}$, then an element of $G(\mathbb{Q})$ is called neat if its image in some faithful representation (equivalently: in every representation) of G is neat. A subgroup of $G(\mathbb{Q})$ is called neat if all its elements are neat. Let $\mathbb{A}_f$ denote the finite adeles of $\mathbb{Q}$. A subgroup of $G(\mathbb{Q})$ is called a congruence subgroup if it is equal to $G(\mathbb{Q}) \cap K$ for some open compact subgroup $K \subset G(\mathbb{A}_f)$.

We then obtain many (smooth) connected mixed Shimura varieties as the quotients of $\mathcal{X}^+$ by $\Gamma_V \rtimes \Gamma$, where Γ is a neat congruence subgroup of $\mathrm{GSp}_{2g}(\mathbb{Q})^+ = \mathrm{GSp}_{2g}(\mathbb{Q}) \cap \mathrm{GSp}_{2g}(\mathbb{R})^+$ and Γ_V is a Γ-invariant lattice in V_{2g}. Each of them comes with a proper holomorphic map to the (smooth) connected pure Shimura variety that is obtained as the quotient of $\mathbb{H}_g$ by Γ. They all have a natural structure of a family of abelian varieties over their associated connected pure Shimura variety (see [139], Corollary 3.12(a) and 3.22(a), and [140], Construction 2.9).

Connected mixed Shimura varieties also carry a canonical algebraic structure over $\bar{\mathbb{Q}}$ with respect to which the Shimura morphisms between them are algebraic and defined over $\bar{\mathbb{Q}}$ (see [139], Proposition 9.24 and Theorem 11.18). With respect to these algebraic structures, the above-mentioned families of abelian varieties become abelian schemes over their associated connected pure Shimura variety. The construction of these algebraic structures uses the fact that some connected mixed Shimura varieties have a canonical moduli interpretation as the universal families over the fine moduli spaces of principally polarized abelian varieties of dimension g with symplectic level l-structure. On these connected mixed Shimura varieties, the canonical algebraic structure is just the one induced by the moduli interpretation (see [139], Corollary 10.10 and Theorem 11.16).

Some remarks are in order here: In [139], mixed Shimura varieties are considered instead of connected mixed Shimura varieties. Now connected mixed Shimura varieties are just the connected components of mixed Shimura varieties (cf. [139], 3.2).

For our particular example of the universal family over the moduli space of principally polarized abelian varieties of dimension g with symplectic level l-structure ($l \geq 4$, l even), we take $\Gamma = G(l) = \{A \in \mathrm{Sp}_{2g}(\mathbb{Z}); A \equiv I_{2g} \mod l\}$ and $\Gamma_V = l\mathbb{Z}^{2g}$. Recall that $\mathbb{A}_f$ denotes the finite adeles of $\mathbb{Q}$ and set $K(l) = \{A \in \mathrm{GSp}_{2g}(\hat{\mathbb{Z}}); A \equiv I_{2g} \mod l\}$.

We have $\mathrm{GL}_{2g}(\mathbb{A}_f) = \mathrm{GL}_{2g}(\mathbb{Q}) \cdot \mathrm{GL}_{2g}(\hat{\mathbb{Z}})$ by Proposition 8.1 in [142]. An element $M \in \mathrm{Sp}_{2g}(\mathbb{A}_f)$ can therefore be written as $M'M''$ with $M' \in \mathrm{GL}_{2g}(\mathbb{Q})$ and $M'' \in \mathrm{GL}_{2g}(\hat{\mathbb{Z}})$. The matrix

$$J = M'^t \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix} M' = M''^{-t} \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix} M''^{-1}$$

lies in $\mathrm{GL}_{2g}(\mathbb{Q}) \cap \mathrm{GL}_{2g}(\hat{\mathbb{Z}}) = \mathrm{GL}_{2g}(\mathbb{Z})$ and is skew-symmetric of determinant 1. Hence, we can find $M''' \in \mathrm{GL}_{2g}(\mathbb{Z})$ such that

$$M'''^t J M''' = \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}.$$

It follows that $M'M''' \in \mathrm{Sp}_{2g}(\mathbb{Q})$ and $M''^{-1}M''' \in \mathrm{Sp}_{2g}(\hat{\mathbb{Z}})$. Therefore, we have $M = (M'M''')(M'''^{-1}M'') \in \mathrm{Sp}_{2g}(\mathbb{Q}) \cdot \mathrm{Sp}_{2g}(\hat{\mathbb{Z}})$. We deduce that

$$\mathrm{GSp}_{2g}(\mathbb{A}_f) = \mathrm{GSp}_{2g}(\mathbb{Q})^+ \cdot \mathrm{Sp}_{2g}(\mathbb{A}_f) \cdot \mathrm{GSp}_{2g}(\hat{\mathbb{Z}}) = \mathrm{GSp}_{2g}(\mathbb{Q})^+ \cdot \mathrm{GSp}_{2g}(\hat{\mathbb{Z}}).$$

Furthermore, the natural homomorphism $\mathrm{Sp}_{2g}(\mathbb{Z}) \to \mathrm{Sp}_{2g}(\mathbb{Z}/l\mathbb{Z})$ is surjective (see [123], Theorem 1). Together, this implies that

$$\left\{ \begin{pmatrix} M_u & 0 \\ 0 & I_g \end{pmatrix}; u \in (\mathbb{Z}/l\mathbb{Z})^* \right\}$$

is a system of representatives for the double quotient $\mathrm{GSp}_{2g}(\mathbb{Q})^+ \backslash \mathrm{GSp}_{2g}(\mathbb{A}_f)/K(l)$, where $M_u \in \mathrm{GSp}_{2g}(\hat{\mathbb{Z}})$ satisfies $M_u \equiv uI_g \mod l$ ($u \in (\mathbb{Z}/l\mathbb{Z})^*$). Similarly, one can show that the corresponding quotient for $V_{2g} \rtimes \mathrm{GSp}_{2g}$ has the same cardinality, where the easier fact that $\mathbb{Z}^{2g} \to (\mathbb{Z}/l\mathbb{Z})^{2g}$ is surjective is needed in addition.

For the moduli interpretation, Pink's definition of symplectic level l-structure in [139], 10.3, includes the choice of a primitive l-th root of unity (corresponding to a choice of u and so a choice of connected component above). In this thesis, we use instead the definition in the Appendix to Chapter 7 of [120], where such a choice is presupposed. Thus, we obtain exactly one connected component of the corresponding mixed Shimura variety. This component is isomorphic to the connected mixed Shimura variety equal to the quotient of $\mathcal{X}^+$ by $l\mathbb{Z}^{2g} \rtimes G(l)$.

This connected mixed Shimura variety is also naturally isomorphic to the quotient of $\mathcal{X}^+$ by $\mathbb{Z}^{2g} \rtimes G(l)$ through the Shimura morphism that sends $(\tau, z) \in \mathbb{H}_g \times \mathbb{C}^g$ to $(\tau, l^{-1}z)$. So this latter connected mixed Shimura variety inherits the same natural moduli interpretation.

In Section 3.2, we will see an explicit quasi-projective immersion, defined through use of the classical theta functions, for the connected mixed Shimura variety that one obtains by choosing $\Gamma = G(l, 2l)$ and $\Gamma_V = \mathbb{Z}^{2g}$ with $l \geq 16$ divisible by 8 and a perfect square (for the definition of $G(l, 2l)$, see Section 3.2). That the image of the map defined through use of the classical theta functions is in fact a quasi-projective variety $\mathfrak{A}_{g,(2l,l)}$ over $\bar{\mathbb{Q}}$ was proven by Mumford, inspired by and building on work of Igusa and others. Mumford also showed that $\mathfrak{A}_{g,(2l,l)}$ has a moduli interpretation as the universal family of principally polarized abelian varieties of dimension g with so-called orthogonal level l-structure (also called level structure "between l and $2l$", level $(l,2l)$-structure, or $(l,\dots,l)$-marking; see [117], §6, and [120], Appendix to Chapter 7, Section B, as well as Sections 3.2 and 3.8). Thanks to the moduli interpretation, we get a morphism (defined over $\bar{\mathbb{Q}}$) from $\mathfrak{A}_{g,(2l,l)}$ to the universal family of principally polarized abelian varieties of dimension g with symplectic level l-structure.

The moduli interpretation associates to the image of $\tau \in \mathbb{H}_g$ the abelian variety $A_\tau = \mathbb{C}^g/\Lambda_\tau$ with $\Lambda_\tau = \tau\mathbb{Z}^g + \mathbb{Z}^g$. The principal polarization is the one induced by the Hermitian form given by $(\operatorname{Im}\tau)^{-1}$ with respect to the standard basis on $\mathbb{C}^g$. Depending on conventions, the symplectic level l-structure is given either by the symplectic basis $l^{-1}\begin{pmatrix}\tau & I_g\end{pmatrix}$ (in [117] and [120]) or by the symplectic basis $l^{-1}\begin{pmatrix}-I_g & \tau\end{pmatrix}$ (in [139]).

A change between conventions corresponds to an automorphism of the universal family. This automorphism is equal to the Shimura morphism $(\tau,z) \mapsto (-\tau^{-1},-\tau^{-1}z)$ that is induced by conjugation by $\left(\begin{pmatrix}0 & I_g\\ -I_g & 0\end{pmatrix},0\right)$ on $V_{2g} \rtimes \mathrm{GSp}_{2g}$.

So, the morphism furnished by the moduli interpretation coincides (up to a Shimura automorphism) with the corresponding Shimura morphism of connected mixed Shimura varieties, i.e. just quotienting out by a larger congruence subgroup. Note that the morphism between the base spaces already determines the morphism between the universal families since principally polarized abelian varieties with symplectic level l-structure have no non-trivial automorphisms for $l \geq 3$. By [2], Exposé XII, Théorème 5.1, the algebraic structure over $\bar{\mathbb{Q}}$ induced by our quasi-projective immersion coincides with the canonical algebraic structure over $\bar{\mathbb{Q}}$ mentioned above. The abelian scheme structures also coincide as they are uniquely determined by the zero section, which is the same in both cases.

In the construction in Section 3.2, we consider a twisted semidirect product of Γ and Γ_V such that $(M,v)(M',v') = (MM', v + M^{-t}v')$ for $M,M' \in \Gamma$ and $v,v' \in \Gamma_V$. We have

$$M^{-t} = \begin{pmatrix}0 & -I_g\\ I_g & 0\end{pmatrix} M \begin{pmatrix}0 & I_g\\ -I_g & 0\end{pmatrix}$$

for all $M \in \mathrm{Sp}_{2g}(\mathbb{Z})$. Accordingly, $v = \begin{pmatrix}v_1\\ v_2\end{pmatrix} \in \mathbb{Z}^{2g}$ acts on $\mathbb{H}_g \times \mathbb{C}^g$ as $\begin{pmatrix}0 & I_g\\ -I_g & 0\end{pmatrix} v$ by sending (τ,z) to $(\tau, z + \tau v_1 + v_2)$. Of course, we obtain the same quotient.

CHAPTER 3

Unlikely intersections between isogeny orbits and curves

> To be, in a word, unborable… It is the key to modern life. If you are immune to boredom, there is literally nothing you cannot accomplish.
>
> D. F. Wallace, *The Pale King*

3.1. Introduction

Let K be a field of characteristic zero, let S be a geometrically irreducible smooth curve, and let $\mathcal{A} \to S$ be an abelian scheme over S of relative dimension g, both defined over K. The structural morphism will be denoted by $\pi : \mathcal{A} \to S$ and is smooth and proper. For any (possibly non-closed) point s of S and any subvariety $\mathcal{V}$ of $\mathcal{A}$, we denote the fiber of $\mathcal{V}$ over s by $\mathcal{V}_s$. The zero section $S \to \mathcal{A}$ is denoted by ϵ.

We fix an algebraic closure $\bar{K}$ of K. All varieties that we consider will be defined over $\bar{K}$, unless explicitly stated otherwise. All varieties will be identified with the set of their closed points over a prescribed algebraic closure of their field of definition. By "irreducible", we will always mean "geometrically irreducible".

We fix an abelian variety A_0 of dimension g and a finite set of $\mathbb{Z}$-linearly independent points $\gamma_1, \dots, \gamma_r$ in A_0. The set can also be empty (i.e. $r = 0$). We define

$$\Gamma = \{\gamma \in A_0;\ \exists N \in \mathbb{N}\colon\ N\gamma \in \mathbb{Z}\gamma_1 + \cdots + \mathbb{Z}\gamma_r\},$$

a subgroup of A_0 of finite rank (and every subgroup of A_0 of finite rank is contained in a group of this form).

The $(g-k)$-enlarged isogeny orbit of Γ (in the family $\mathcal{A}$) is defined as

$$\begin{aligned}\mathcal{A}_\Gamma^{[k]} = \{p \in \mathcal{A}_s;\ s \in S,\ \exists \phi : A_0 \to \mathcal{A}_s \text{ isogeny and an abelian subvariety}\\ B_0 \subset A_0 \text{ of codimension} \geq k \text{ such that } p \in \phi(\Gamma + B_0)\}.\end{aligned} \tag{3.1.1}$$

This condition is equivalent to the existence of an isogeny $\psi : \mathcal{A}_s \to A_0$ with $\psi(p) \in \Gamma + B_0$. The isogeny orbit of Γ is defined as $\mathcal{A}_\Gamma = \mathcal{A}_\Gamma^{[g]}$.

Let ξ be the generic point of S. We fix an algebraic closure $\overline{K(S)}$ of $\bar{K}(S)$ and let $\left(\mathcal{A}_\xi^{\overline{K(S)}/\bar{K}}, \mathrm{Tr}\right)$ denote the $\overline{K(S)}/\bar{K}$-trace of $\mathcal{A}_\xi$, as defined in Chapter VIII, §3 of [81], where we consider $\mathcal{A}_\xi$ as a variety over $\overline{K(S)}$ by abuse of notation. We

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call $\mathcal{A}$ isotrivial if Tr is surjective. In this chapter, we investigate the following conjecture, a slightly modified version of Gao's Conjecture 1.2, which he calls the André-Pink-Zannier conjecture, in [51].

CONJECTURE 3.1.1. *(Modified André-Pink-Zannier over a curve) Suppose that $\mathcal{A} \to S$ is not isotrivial. Let $\mathcal{V} \subset \mathcal{A}$ be an irreducible subvariety. If $\mathcal{A}_\Gamma \cap \mathcal{V}$ is Zariski dense in $\mathcal{V}$, then one of the following two conditions is satisfied:*

(i) The variety $\mathcal{V}$ is a translate of an abelian subvariety of $\mathcal{A}_s$ by a point of $\mathcal{A}_\Gamma \cap \mathcal{A}_s$ for some $s \in S$.
(ii) We have $\pi(\mathcal{V}) = S$ and over $\overline{K(S)}$, every irreducible component of $\mathcal{V}_\xi$ is a translate of an abelian subvariety of $\mathcal{A}_\xi$ by a point in $\mathrm{Tr}\left(\mathcal{A}_\xi^{\overline{K(S)}/\bar{K}}(\bar{K})\right) + (\mathcal{A}_\xi)_{\mathrm{tors}}$.

We need to formulate the conclusion in this somewhat involved manner in order to account for the fact that there can exist abelian subvarieties of $\mathcal{A}_\xi$ and points in $(\mathcal{A}_\xi)_{\mathrm{tors}}$ that are not defined over $\bar{K}(S)$ and that the morphism Tr is not necessarily defined over $\bar{K}(S)$. It can be considered one relative version of the Mordell-Lang conjecture, proven for abelian varieties by Vojta [186], Faltings [44], and Hindry [72], and in its most general form by McQuillan in [110], in analogy to the relative Manin-Mumford results proven by Masser and Zannier in e.g. [103]. As we can always assume that K is finitely generated over $\mathbb{Q}$ and then embed it in $\mathbb{C}$, it suffices to prove the conjecture for subfields of $\mathbb{C}$.

Prima facie, Gao's conjecture only concerns irreducible subvarieties of the universal family of principally polarized abelian varieties of fixed dimension and fixed sufficiently large level structure. However, we can assume without loss of generality that $\mathcal{A}$ is contained in a suitable universal family $\mathfrak{A}_{g,(2l,l)}$ corresponding to principally polarized abelian varieties of dimension g with so-called orthogonal level l-structure (cf. Sections 3.2 and 3.8), which reduces Conjecture 3.1.1 to the case considered by Gao. The condition that the base S in this situation is a weakly special curve in the moduli space seems to be missing in our formulation of the conjecture, but it follows directly from Orr's Theorem 1.2 in [124] that Conjecture 3.1.1 can be further reduced to this case. The conjecture is stronger than Gao's in that it involves a subgroup of rank possibly larger than 1 and does not demand that the isogenies are polarized. It is weaker in that the base variety S is assumed to be a curve.

Gao showed in Section 8 of [51] that Conjecture 3.1.1 follows from Pink's Conjecture 1.6 in [140] in the more general setting of generalized Hecke orbits in mixed Shimura varieties, where it is enough to assume Pink's conjecture for all fibered powers of universal families of principally polarized abelian varieties of fixed dimension and fixed, sufficiently large level structure. By Theorem 3.3 in [141], Conjecture 1.6 in [140] is a consequence of Pink's even more general Conjecture 1.1 in [141] on unlikely intersections in mixed Shimura varieties. If Γ has rank zero, Conjecture 3.1.1 is contained in a special-point conjecture of Zannier (see [51], Conjecture 1.4).

Progress towards Conjecture 3.1.1 has only been made if $\mathcal{V} = \mathcal{C}$ is a curve or if the rank of Γ is zero. Furthermore, many results are confined to the case where K is a number field. Lin and Wang have proven the conjecture for K a number field, $\mathcal{V}$ a curve, Γ finitely generated, and A_0 simple (Theorem 1.1 in [88]). Habegger has proven it for K a number field, Γ of rank zero, and $\mathcal{A}$ a fibered power of a non-isotrivial elliptic scheme (Theorem 1.2 in [68]). Pila has proven it for arbitrary K, Γ of rank zero, and $\mathcal{A}$ inside a product of elliptic modular surfaces (Theorem 6.2 in [133]). Gao has proven it for arbitrary K and Γ of rank zero (Theorem 1.5 in [51]) as well as for arbitrary K, $\mathcal{V}$ a curve, and Γ of rank at most one, but in this case he has to fix polarizations of A_0 and $\mathcal{A}$ and assume that the isogenies are polarized (Theorem 1.6 in [51]).

From now on, we will always assume that $K \subset \mathbb{C}$ is a number field and take as $\bar{K} = \bar{\mathbb{Q}}$ its algebraic closure in $\mathbb{C}$. We expect however that Theorem 3.1.3 can be generalized to the transcendental case in the same way as Gao's by use of the Moriwaki height instead of the Weil height together with specialization arguments (see Section 5.1 for more details).

The purpose of this chapter is twofold: First, we prove Conjecture 3.1.1 in Theorem 3.1.3 if K is a number field and $\mathcal{V} = \mathcal{C}$ is a curve. Second, we investigate what happens when $\mathcal{C} \cap \mathcal{A}_\Gamma^{[k]}$ is infinite for some arbitrary $k \in \{0, \ldots, g\}$. Here, the case $k = g$ corresponds to Conjecture 3.1.1. If $k < g$, the condition is weaker (if $k = 0$, it is void), so we expect a weaker conclusion. We prove the strongest possible conclusion in Theorem 3.1.2, of which Theorem 3.1.3 thus becomes a special case.

The problem of intersecting a fixed subvariety with algebraic subgroups originates in works of Bombieri-Masser-Zannier [22] and Zilber [195] for powers of the multiplicative group. The analogous problem in a fixed abelian variety has also been the object of much study; we just mention the work of Habegger and Pila [70], from which we use several results in our proof. The intersection of a subvariety of a fixed abelian variety with translates of abelian subvarieties by points of a subgroup of finite rank has been studied by Rémond in e.g. [156]. While there has been intensive study of unlikely intersections between a curve in an abelian scheme and flat algebraic subgroup schemes, culminating in the article by Barroero and Capuano [12], ours seems to be the first result that combines intersecting with positive-dimensional algebraic subgroups with an isogeny condition on the fiber.

We can now state our main results. Recall that S is a smooth irreducible curve and $\mathcal{A} \to S$ is an abelian scheme, both defined over K, while $\mathcal{C} \subset \mathcal{A}$ is a closed irreducible curve, defined over $\bar{\mathbb{Q}}$, A_0 is an abelian variety defined over $\bar{\mathbb{Q}}$, $\gamma_1, \ldots, \gamma_r \in A_0(\bar{\mathbb{Q}})$, and $\Gamma \subset A_0$ is the subgroup of all $\gamma \in A_0$ such that $N\gamma \in \mathbb{Z}\gamma_1 + \cdots + \mathbb{Z}\gamma_r$ for some $N \in \mathbb{N}$.

THEOREM 3.1.2. *Suppose that $\mathcal{A} \to S$ is not isotrivial. If $\mathcal{A}_\Gamma^{[k]} \cap \mathcal{C}$ is infinite and $\pi(\mathcal{C}) = S$, then $\mathcal{C}$ is contained in an irreducible subvariety $\mathcal{W}$ of $\mathcal{A}$ of codimension $\geq k$ with the following property: Over $\overline{\bar{\mathbb{Q}}(S)}$, every irreducible component of $\mathcal{W}_\xi$ is a translate of an abelian subvariety of $\mathcal{A}_\xi$ by a point in $(\mathcal{A}_\xi)_{\mathrm{tors}} + \mathrm{Tr}\left(\mathcal{A}_\xi^{\overline{\bar{\mathbb{Q}}(S)}/\bar{\mathbb{Q}}}(\bar{\mathbb{Q}})\right)$.*

THEOREM 3.1.3. *Suppose that $\mathcal{A} \to S$ is not isotrivial. If $\mathcal{A}_\Gamma \cap \mathcal{C}$ is infinite, then one of the following two conditions is satisfied:*

(i) The curve $\mathcal{C}$ is a translate of an abelian subvariety of $\mathcal{A}_s$ by a point of $\mathcal{A}_\Gamma \cap \mathcal{A}_s$ for some $s \in S$.

(ii) The zero-dimensional variety $\mathcal{C}_\xi$ is contained in $(\mathcal{A}_\xi)_{\mathrm{tors}} + \mathrm{Tr}\left(\mathcal{A}_\xi^{\overline{\bar{\mathbb{Q}}(S)}/\bar{\mathbb{Q}}}(\bar{\mathbb{Q}})\right)$.

From Theorem 3.1.3, we can deduce the following corollary:

COROLLARY 3.1.4. *Let $A_{g,(2l,l)}$ be the moduli space of principally polarized abelian varieties of dimension g with orthogonal level l-structure as defined in Section 3.2 and l sufficiently large and let A and B be abelian varieties with* $\dim B = g$. *Let $C \subset A_{g,(2l,l)} \times A$ be a closed irreducible curve and let* $\mathrm{pr}_1 : C \to A_{g,(2l,l)}$ *and* $\mathrm{pr}_2 : C \to A$ *be the canonical projections. Let $\Gamma' \subset A$ be a subgroup of finite rank and let $\Sigma \subset A_{g,(2l,l)}$ be the set of $s \in A_{g,(2l,l)}$ corresponding to abelian varieties that are isogenous to B. If $C \cap (\Sigma \times \Gamma')$ is infinite, then either* pr_1 *or* pr_2 *is constant.*

We thereby prove Conjecture 1.7 of Buium and Poonen in [27]: If S is a modular curve or a Shimura curve, then a Zariski open subset S' of S has a moduli interpretation which yields a quasi-finite forgetful modular morphism from S' to the coarse moduli space A_g of principally polarized abelian varieties of dimension $g \in \{1, 2\}$. Similarly, we have a quasi-finite morphism $A_{g,(2l,l)} \to A_g$. We can then form the curve $S' \times_{A_g} A_{g,(2l,l)}$, which admits quasi-finite morphisms to S' and $A_{g,(2l,l)}$, and reduce the conjecture to Corollary 3.1.4. The conjecture of Buium and Poonen has been proven independently by Baldi in [8] through the use of equidistribution results. He was also able to replace Γ' by a fattening Γ'_ϵ for some $\epsilon > 0$ (see [8] for the definition of Γ'_ϵ). Such an extension seems to lie outside the reach of our methods though.

The proof of Theorem 3.1.2 uses point counting and o-minimality and in particular a later refinement of the theorem of Pila-Wilkie on rational points on definable sets in [137]. In applying this result to problems of unlikely intersections in diophantine geometry, we follow the standard strategy as devised by Zannier for the new proof of the Manin-Mumford conjecture by Pila and him in [138]. It is described in Zannier's book [193]. In Section 3.2, we introduce some notation and make several reduction steps.

In Sections 3.3 and 3.4, we bound the "height" of all important quantities from above in terms of the degree of the varying point $p = \phi(q) \in \mathcal{A}_\Gamma^{[k]} \cap \mathcal{C}$ over the fixed number field K. The main new ideas of this chapter are to be found in these two sections. In order to treat non-polarized isogenies, we extend a result by Orr to show that the isogeny ϕ between A_0 and $\mathcal{A}_s$ can be chosen such that certain associated quantities are bounded in the required way; first of all, we apply the isogeny theorem of Masser-Wüstholz to show that the degree of the isogeny can be bounded in this way. As a consequence of our extension of Orr's result we can then bound the height of q for this choice of ϕ. (After maybe enlarging Γ, we can fix for each $s \in S$ such that A_0 and $\mathcal{A}_s$ are isogenous one choice of isogeny; see Lemma 3.2.2.)

We bound the degree of the smallest translate of an abelian subvariety of A_0^{r+1} by a torsion point that contains $(q, \gamma_1, \dots, \gamma_r)$ through an application of a proposition by Habegger and Pila. Using this and a lemma of Rémond, we can then write $q = \gamma + b$ with $\gamma \in \Gamma$ of controlled height and b in an abelian subvariety of controlled codimension and degree. If N is the smallest natural number such that $N\gamma \in \bigoplus_{i=1}^r \mathbb{Z}\gamma_i$, we finally bound N by applying a lemma of Habegger and Pila, some elementary diophantine approximation, and lower height bounds on abelian varieties due to Masser.

In Section 3.5, we give a brief introduction to o-minimal structures in as much depth as is necessary to state a variant of the Pila-Wilkie theorem, due to Habegger and Pila, on "semirational" points of bounded height.

In Section 3.6, the definability in a suitable o-minimal structure of the analytic uniformization map associated to our abelian scheme is shown, when restricted to some fundamental domain, by use of a theorem of Peterzil-Starchenko from [128]. In Section 3.7, we record the necessary algebraic independence result of "logarithmic Ax" type by Gao in [52], which generalizes work by André in [4] and by Bertrand in [16].

Finally, we put all the pieces together in Section 3.8 and prove Theorem 3.1.2, Theorem 3.1.3, and Corollary 3.1.4.

3.2. Preliminaries and notation

For our proof of Theorem 3.1.3, we will restrict ourselves in the following sections to subfamilies of the universal family $\mathfrak{A}_{g,(2l,l)} \to A_{g,(2l,l)}$ of principally polarized abelian varieties with so-called orthogonal level l-structure for a natural number $l \geq 16$ which is divisible by 8 and a perfect square, and identify π and ϵ with the natural projection and zero section of that family. If $\mathbb{H}_g$ denotes the Siegel upper half space in dimension g (i.e. symmetric matrices in $\mathrm{M}_g(\mathbb{C})$ with positive definite imaginary part), then $\mathfrak{A}_{g,(2l,l)}$ is a quotient of $\mathbb{H}_g \times \mathbb{C}^g$ by the semidirect product of the congruence subgroup

$$G(l,2l) = \left\{ M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_{2g}(\mathbb{Z});\ M \equiv I_{2g} \mod l, \right.$$
$$\left. \mathrm{diag}(AB^t) \equiv \mathrm{diag}(CD^t) \equiv 0 \mod 2l \right\}$$

of $\mathrm{Sp}_{2g}(\mathbb{Z})$ with $\mathbb{Z}^{2g}$, where diag denotes the diagonal of a matrix. We will show at the end of our work in Section 3.8 how to deduce the result for arbitrary families.

The group $G(l,2l)$ is the same as the group $\Gamma(l,2l)$ defined on p. 422 of [101]. Let $G_{lI_g}(lI_g)_0$ be defined as in Section 8.9 of [19]. Then there is an isomorphism from $G(l,2l)$ to $G_{lI_g}(lI_g)_0$ given by sending M to $\begin{pmatrix} I_g & 0 \\ 0 & l^{-1}I_g \end{pmatrix} M \begin{pmatrix} I_g & 0 \\ 0 & lI_g \end{pmatrix}$; see [19], Section 8.8 and 8.9, and note that l is even.

The group law on the semidirect product is given by $(M',z')(M,z) = (M'M, z' + (M')^{-t}z)$ and the action of the group is given by

$$\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}, \begin{pmatrix} m \\ n \end{pmatrix} \right)(\tau, z) = (\tau', (C\tau + D)^{-t}z + \tau' m + n),$$

where

$$\tau' = \begin{pmatrix} A & B \\ C & D \end{pmatrix}[\tau] := (A\tau + B)(C\tau + D)^{-1}.$$

The action of course extends to an action of $\mathrm{Sp}_{2g}(\mathbb{R}) \ltimes \mathbb{R}^{2g}$ (with the same group law) and then also restricts to the usual action of $\mathrm{Sp}_{2g}(\mathbb{R})$ on $\mathbb{H}_g$. If $M \in \mathrm{Sp}_{2g}(\mathbb{R})$ and $\tau \in \mathbb{H}_g$, we will denote this last action as above by $M[\tau]$ to avoid confusion with ordinary matrix multiplication.

By applying Proposition 8.2.5 from [19] and Cartan's Exposé 11 in Volume 2 of [1], we see that our universal family is a complex analytic space because the group action is proper and discontinuous; Proposition 8.2.5 in [19] only says that the action of $G(l,2l)$ on $\mathbb{H}_g$ is proper and discontinuous, but this quickly implies the same for the action of its semidirect product with $\mathbb{Z}^{2g}$ on $\mathbb{H}_g \times \mathbb{C}^g$. However, the universal family is in fact a quasi-projective variety, defined over $\mathbb{Q}$. In the following proposition, we recall some well-known facts about it.

PROPOSITION 3.2.1. *There exist holomorphic maps*

$$\exp : \mathbb{H}_g \times \mathbb{C}^g \to \mathbb{P}^{l^g-1}(\mathbb{C}) \times \mathbb{P}^{l^g-1}(\mathbb{C})$$

and $\iota : \mathbb{H}_g \to \mathbb{P}^{l^g-1}(\mathbb{C})$ *with the following properties:*

(i) There is a commutative diagram

$$\begin{array}{ccc} \mathbb{H}_g \times \mathbb{C}^g & \xrightarrow{\exp} & \mathbb{P}^{l^g-1}(\mathbb{C}) \times \mathbb{P}^{l^g-1}(\mathbb{C}) \\ \downarrow & & \downarrow \\ \mathbb{H}_g & \xrightarrow{\quad\iota\quad} & \mathbb{P}^{l^g-1}(\mathbb{C}) \end{array},$$

where the vertical maps are projections to the first factor.

(ii) We have $\exp(\tau,z) = \exp(\tau',z')$ *if and only if* (τ,z), (τ',z') *lie in the same* $G(l,2l) \ltimes \mathbb{Z}^{2g}$*-orbit and* $\exp$ *descends to an analytic embedding of the quotient. Similarly, we have* $\iota(\tau) = \iota(\tau')$ *if and only if* τ, τ' *lie in the same* $G(l,2l)$*-orbit and* ι *descends to an analytic embedding of the quotient.*

(iii) The images $\exp(\mathbb{H}_g \times \mathbb{C}^g)$ *and* $\iota(\mathbb{H}_g)$ *are locally closed with respect to the Zariski topology in* $\mathbb{P}^{l^g-1}(\mathbb{C}) \times \mathbb{P}^{l^g-1}(\mathbb{C})$ *and* $\mathbb{P}^{l^g-1}(\mathbb{C})$ *respectively. They are irreducible smooth varieties, defined over* $\mathbb{Q}$*.*

(iv) $\exp(\mathbb{H}_g \times \mathbb{C}^g) \to \iota(\mathbb{H}_g)$ *is an abelian scheme, defined over* $\mathbb{Q}$*, with zero section* $p \mapsto (p,p)$*.*

(v) $\exp(\tau,\cdot)$ *is a surjective group homomorphism from* $\mathbb{C}^g$ *to* $\exp(\{\tau\} \times \mathbb{C}^g)$ *with kernel* $\Omega_\tau \mathbb{Z}^{2g}$*, where*

$$\Omega_\tau = \begin{pmatrix} \tau & I_g \end{pmatrix}.$$

(vi) The very ample line bundle on $\exp(\{\tau\} \times \mathbb{C}^g)$ *that is induced by this embedding is the* l*-th tensor power of a symmetric ample line bundle. Under the uniformization* $\exp(\{\tau\} \times \mathbb{C}^g) \simeq \mathbb{C}^g/\Omega_\tau\mathbb{Z}^{2g}$ *given by* $\exp$*, the Hermitian form on* $\mathbb{C}^g$ *induced by this second line bundle is given by the matrix* $(\operatorname{Im}\tau)^{-1}$*.*

PROOF. We can explicitly give the maps, using the classical theta functions. For this, we define

$$\theta[a,b](\tau,z) = \sum_{m \in \mathbb{Z}^g} \exp(\pi\sqrt{-1}(m+a)^t\tau(m+a) + 2\pi\sqrt{-1}(m+a)^t(z+b))$$

for $\tau \in \mathbb{H}_g$, $z \in \mathbb{C}^g$, and $a,b \in \mathbb{Q}^g$. The series defines a holomorphic function by Proposition 8.5.4 in [19]. For $c \in \mathbb{Q}^g$ and $(\tau,z) \in \mathbb{H}_g \times \mathbb{C}^g$, we put

$$\theta_c(\tau,z) = \theta[c,0](\tau,z).$$

We then define

$$\phi(\tau, z) = [\theta_{c_0}(\tau, z) : \cdots : \theta_{c_{l^g-1}}(\tau, z)]$$

and $\iota(\tau) = \phi(l\tau, 0)$ as well as

$$\exp(\tau, z) = (\phi(l\tau, 0), \phi(l\tau, lz)),$$

where the c_i run over the set $\{0, \frac{1}{l}, \ldots, 1 - \frac{1}{l}\}^g$ ($i = 0, \ldots, l^g - 1$).

Property (i) now follows directly from the definitions. For property (ii), we note that $\iota(\tau) = \iota(\tau')$ if and only if τ and τ' lie in the same $G(l, 2l)$-orbit by Proposition 8.10.2 in [19]. In order to see this, one has to verify that τ and τ' lie in the same $G(l, 2l)$-orbit if and only if $l\tau$ and $l\tau'$ lie in the same $G_{lI_g}(lI_g)_0$-orbit. This holds thanks to the above-mentioned isomorphism between $G_{lI_g}(lI_g)_0$ and $G(l, 2l)$.

Suppose now that $\tau' = M[\tau]$ with $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in G(l, 2l)$ and that $\exp(\tau', z') = \exp(\tau, z)$. By the Theta Transformation Formula 8.6.1 and the proof of Lemma 8.9.2 in [19], we have $\exp(\tau, z) = \exp(\tau', (C\tau + D)^{-t}z)$ (again some yoga between $G(l, 2l)$ and $G_{lI_g}(lI_g)_0$ is necessary). So we can reduce to the case $\tau = \tau'$. By Remark 8.5.3(d) in [19] and the Theorem of Lefschetz (4.5.1 in [19]), the map $\phi(l\tau, \cdot)$ is an embedding of $\mathbb{C}^g/(l\mathbb{Z}^g + l\tau\mathbb{Z}^g)$ into projective space (note that $l \geq 3$). We deduce that $lz - lz' \in l\Omega_\tau\mathbb{Z}^{2g}$ and therefore $z - z' \in \Omega_\tau\mathbb{Z}^{2g}$. Reversing the arguments shows the other direction of the implication.

Note that $M[\tau] = \tau$ implies $M = I_{2g}$ by our choice of l: As the action is proper and discontinuous, we certainly have $M[\tau] = \tau$ for only finitely many M. So these M form a finite subgroup of $\mathrm{GL}_{2g}(\mathbb{Z})$. By a result of Minkowski (see [113], §1), reduction modulo 4 is injective when restricted to any finite subgroup of $\mathrm{GL}_{2g}(\mathbb{Z})$. Since 4 divides l, the same holds for l and hence M can only be the identity matrix. Thus, the action is free as well and the quotient map is a covering.

The images $\iota(\mathbb{H}_g)$ and $\exp(\mathbb{H}_g \times \mathbb{C}^g)$ are Zariski open subsets of irreducible subvarieties in $\mathbb{P}^{l^g-1}(\mathbb{C})$ and $\mathbb{P}^{l^g-1}(\mathbb{C}) \times \mathbb{P}^{l^g-1}(\mathbb{C})$ respectively of respective dimensions $\frac{g(g+1)}{2}$ and $\frac{g(g+1)}{2} + g$ (see [75] and [117]). By Theorem 8.4.4 in [59], the quotient $G(l, 2l)\backslash\mathbb{H}_g$ maps biholomorphically onto an open subset of the normalization of the closure of $\iota(\mathbb{H}_g)$. Over each point in $\iota(\mathbb{H}_g)$ lies exactly one point of the normalization by [76]. Hence, the induced map from $G(l, 2l)\backslash\mathbb{H}_g$ to $\iota(\mathbb{H}_g)$ is proper and therefore a homeomorphism. The same follows for the map from $(G(l, 2l) \ltimes \mathbb{Z}^{2g})\backslash(\mathbb{H}_g \times \mathbb{C}^g)$ to $\exp(\mathbb{H}_g \times \mathbb{C}^g)$ as $(G(l, 2l) \ltimes \mathbb{Z}^{2g})\backslash(\mathbb{H}_g \times \mathbb{C}^g)$ is proper over $G(l, 2l)\backslash\mathbb{H}_g$. To show that the induced maps on the quotients are analytic embeddings, it therefore suffices to show that the differentials of ι and exp are injective, and this follows from Proposition 8.10.3 and Theorem 4.5.1 in [19].

For property (iii), we know thanks to Lemma 3.1 in [101], due to Igusa and Mumford, that there are polynomial equations with integer coefficients that define $\exp(\mathbb{H}_g \times \mathbb{C}^g)$ over the base $\iota(\mathbb{H}_g)$. That $\iota(\mathbb{H}_g)$ is locally closed in the Zariski topology and defined over $\mathbb{Q}$ follows from the discussion in [101], p. 415, and the references given there. Finally, $\exp(\mathbb{H}_g \times \mathbb{C}^g)$ and $\iota(\mathbb{H}_g)$ are smooth and irreducible since they are analytically isomorphic to quotients of connected complex analytic manifolds under a free and properly discontinuous group action.

For property (iv), we note that the projection morphism is proper and $\exp(\mathbb{H}_g \times \mathbb{C}^g)$ and $\iota(\mathbb{H}_g)$ are smooth. Furthermore, every fiber is analytically isomorphic to an abelian variety $\mathbb{C}^g/\Omega_\tau\mathbb{Z}^{2g}$ since $\exp(\tau, z) = \exp(\tau, z')$ if and only if $z - z' \in \Omega_\tau\mathbb{Z}^{2g}$

by the proof of (ii) above. Especially, all fibers have the same dimension and are smooth, so the projection morphism is smooth as well. For the algebraicity of the induced group structure, we refer to Lemmata 3.4 and 3.5 in [101], where suitable polynomial addition and inversion formulae with integer coefficients are constructed for the theta functions. This also proves property (v).

Property (vi) follows by computing the factor of automorphy of the embedding $\exp(\tau,\cdot)$ of $\mathbb{C}^g/\Omega_\tau\mathbb{Z}^{2g}$: An elementary computation shows that

$$\theta_c(l\tau, l(z+m+\tau n)) = \exp(-\pi\sqrt{-1}ln^t\tau n - 2\pi\sqrt{-1}ln^t z)\theta_c(l\tau, lz)$$

for all $c \in \{0, \frac{1}{l}, \ldots, 1-\frac{1}{l}\}^g$ and all $m, n \in \mathbb{Z}^g$. By Remark 8.5.3(d) in [19], this factor of automorphy belongs to the l-th tensor power of a symmetric ample line bundle that under the given uniformization is associated to the Hermitian form given by $(\operatorname{Im}\tau)^{-1}$ on $\mathbb{C}^g$. □

Using the proposition, we may identify $\exp(\mathbb{H}_g \times \mathbb{C}^g)$ and $\iota(\mathbb{H}_g)$ with $\mathfrak{A}_{g,(2l,l)}(\mathbb{C})$ and $A_{g,(2l,l)}(\mathbb{C})$ and use $\mathfrak{A}_{g,(2l,l)}$ and $A_{g,(2l,l)}$ for the corresponding quasi-projective varieties, defined over $\mathbb{Q}$. We will denote the Zariski closures in $\mathbb{P}^{l^g-1}$ and $\mathbb{P}^{l^g-1} \times \mathbb{P}^{l^g-1}$ of these varieties by $\overline{\mathfrak{A}_{g,(2l,l)}}$ and $\overline{A_{g,(2l,l)}}$ respectively; these are (usually highly singular) projective varieties, also defined over $\mathbb{Q}$. The projection from $\mathbb{P}^{l^g-1} \times \mathbb{P}^{l^g-1}$ onto the first factor yields a morphism $\pi : \overline{\mathfrak{A}_{g,(2l,l)}} \to \overline{A_{g,(2l,l)}}$. The embedding from the proposition yields very ample line bundles $\mathcal{L}$ on $\overline{\mathfrak{A}_{g,(2l,l)}}$ and L on $\overline{A_{g,(2l,l)}}$.

From now on, we assume that $S \subset A_{g,(2l,l)}$ is an irreducible, smooth, locally closed curve (not necessarily closed in $A_{g,(2l,l)}$), $\mathcal{A} = \pi^{-1}(S)$, and $\mathcal{C} \subset \mathcal{A}$ is an irreducible closed curve. We denote by $\overline{\mathcal{C}}$ and $\overline{S}$ the Zariski closures of $\mathcal{C}$ and S in $\overline{\mathfrak{A}_{g,(2l,l)}}$ and $\overline{A_{g,(2l,l)}}$ respectively. The abelian scheme $\mathcal{A} \to S$ and the curve $\overline{S}$ are defined over K.

After maybe enlarging K, we can and will assume without loss of generality that A_0, the addition morphism $A_0 \times A_0 \to A_0$, the inversion morphism $A_0 \to A_0$, $\mathcal{C}$, and $\overline{\mathcal{C}}$ are defined over K and that A_0 is principally polarized. For this, we might have to replace A_0 by an isogenous abelian variety and Γ by its pre-image under the corresponding isogeny. This does not change the isogeny orbit, so does not change the statement we want to prove.

We fix a symmetric ample line bundle L_0 which gives us a principal polarization on A_0 and fix once and for all a uniformization $\mathbb{C}^g/\Omega_{\tau_0}\mathbb{Z}^{2g}$ of $A_0(\mathbb{C})$ such that the Hermitian form on $\mathbb{C}^g$ associated to L_0 is given by $(\operatorname{Im}\tau_0)^{-1}$, $\Omega_{\tau_0} = (\, \tau_0 \; I_g \,)$, and τ_0 lies in the Siegel fundamental domain (see Definition 3.3.2). We denote the corresponding map $\mathbb{C}^g \to A_0(\mathbb{C})$ by $\exp_0$. Using Weil's Height Machine (see [73], Theorem B.3.2 and B.3.6), we also get a (logarithmic projective) height $h_{A_0} = h_{A_0,L_0}$ on A_0. With the usual construction due to Néron and Tate (see [73], Theorem B.5.1) we then obtain a canonical height $\widehat{h}_{A_0}$ on A_0.

After maybe enlarging K again, we can assume that L_0 is defined over K, $\gamma_1, \ldots, \gamma_r \in A_0(K)$, and every endomorphism of A_0 is defined over K. Since the endomorphism ring of A_0 is finitely generated as a $\mathbb{Z}$-module, we may assume that Γ is mapped into itself by every endomorphism of A_0 by enlarging Γ if necessary (which only makes Theorems 3.1.2 and 3.1.3 stronger). We will generally assume that $r \geq 1$ for simplicity; one can either ensure this by enlarging Γ and K or one can check that our proof also works *mutatis mutandis* if $r = 0$.

The line bundle L restricts to a very ample line bundle $L_{\overline{S}}$ on $\overline{S}$. For each $s \in S$, the restriction of $\mathcal{L}$ to $\mathcal{A}_s$ is a very ample symmetric line bundle $\mathcal{L}_s$ by Proposition 3.2.1(vi). From the embeddings into projective space by theta functions, we directly obtain associated heights $h_{\overline{S}}$ on $\overline{S}$ and h_s on $\mathcal{A}_s$ ($s \in S$) as well as a canonical height $\widehat{h}_s$ on $\mathcal{A}_s$.

The following technical lemma shows that for each $s \in S$ we can fix an isogeny ϕ_s in the definition of $\mathcal{A}_\Gamma^{[k]}$.

LEMMA 3.2.2. *For each $s \in S$ such that $\mathcal{A}_s$ and A_0 are isogenous, fix an isogeny $\phi_s : A_0 \to \mathcal{A}_s$. For Γ as above, we have*

$$\begin{aligned} \mathcal{A}_\Gamma^{[k]} = \{p \in \mathcal{A}_s;\ s \in S,\ \mathcal{A}_s \textit{ and } A_0 \textit{ isogenous, and there exists an abelian} \\ \textit{subvariety } B_0 \subset A_0 \textit{ of codimension} \geq k \textit{ such that } p \in \phi_s(\Gamma + B_0)\}. \end{aligned} \tag{3.2.1}$$

PROOF. We prove the non-trivial inclusion "$\subset$". Suppose that $p \in \mathcal{A}_\Gamma^{[k]}$. Then p lies in some $\mathcal{A}_s$ ($s \in S$) such that $\mathcal{A}_s$ and A_0 are isogenous. By definition, there is an isogeny $\phi : A_0 \to \mathcal{A}_s$, an abelian subvariety B_0 of A_0 of codimension $\geq k$, and $\gamma \in \Gamma$ such that $p \in \phi(\gamma + B_0)$.

We denote by $\tilde{\phi}_s$ the isogeny from $\mathcal{A}_s$ to A_0 such that $\phi_s \circ \tilde{\phi}_s$ is multiplication by $\deg \phi_s$ on $\mathcal{A}_s$. Then $\chi = \tilde{\phi}_s \circ \phi$ is an endomorphism of A_0 and $\phi_s \circ \chi = (\deg \phi_s)\phi$.

We choose $\tilde{\gamma} \in A_0$ with $(\deg \phi_s)\tilde{\gamma} = \gamma$ and get

$$p \in \phi(\gamma + B_0) = \phi((\deg \phi_s)\tilde{\gamma}) + \phi((\deg \phi_s)B_0) = \phi_s(\chi(\tilde{\gamma})) + \phi_s(\chi(B_0)).$$

We show that $\chi(\tilde{\gamma}) \in \Gamma$ (so $p \in \phi_s(\Gamma) + \phi_s(\chi(B_0))$). Since $(\deg \phi_s)\tilde{\gamma} = \gamma \in \Gamma$, it follows that $\tilde{\gamma}$ lies in Γ as well. By our assumption above, Γ is mapped into itself by χ. Hence, $\chi(\tilde{\gamma})$ belongs to Γ as desired and the lemma follows since $\chi(B_0)$ is again an abelian subvariety of A_0 of codimension $\geq k$. $\square$

We take ϕ_s as an isogeny between $\mathcal{A}_s$ and A_0 of minimal degree, i.e. there exists no isogeny $\psi : A_0 \to \mathcal{A}_s$ of degree less than $\deg \phi_s$. By Théorème 1.4 of Gaudron-Rémond in [56], which improves a theorem of Masser-Wüstholz ([100], p. 460), there exist constants c_{MW} and κ_{MW}, depending only on A_0, such that

$$\deg \phi_s \leq c_{MW}[K(s) : K]^{\kappa_{MW}}, \tag{3.2.2}$$

independently of s. Note that $\mathcal{A}_s$ and A_0 are both defined over $K(s)$.

3.3. Height bounds for isogenies

In the previous section, we took as ϕ_s just any isogeny between A_0 and $\mathcal{A}_s$ of minimal degree. This is fine in the case of elliptic curves, but in arbitrary dimension, we have to pick the distinguished isogeny more carefully. This will be achieved in Proposition 3.3.3 and Corollary 3.3.4, where we replace ϕ_s by $\phi_s \circ \sigma$ for some well-chosen automorphism σ of A_0.

Proposition 3.3.3(ii) and Corollary 3.3.4(ii) are essentially contained in Orr's work [124], albeit formulated rather differently, and our proofs of these results basically run along the same lines as his. Another way to get the desired bounds on quantities associated to an isogeny between A_0 and $\mathcal{A}_s$ ($s \in S$) would be to replace the use of Orr's Proposition 4.2 from [124] with the endomorphism estimate from Lemma 5.1 of Masser and Wüstholz in [102] for $A_0 \times \mathcal{A}_s$ (an improved, completely explicit bound can be deduced from Section 9 of [56], Lemme 2.11 in [158], and Minkowski's second theorem) and an argument as in Section 6 of [102]. Afterwards, one could continue as we do here and obtain bounds that are polynomial (in the sense of (3.3.1)) not necessarily in the degree of the isogeny, but certainly in $[K(s):K]$.

Before we can prove the proposition, we need the following technical lemma:

LEMMA 3.3.1. *Let g be a natural number and $M \in \mathrm{M}_{2g}(\mathbb{Z})$ with $\det M \neq 0$. Let*

$$\mathcal{H} = H\left(M^t \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix} M\right).$$

Then there are constants $C = C(g)$ and $\kappa = \kappa(g)$ and matrices $Q \in \mathrm{Sp}_{2g}(\mathbb{Z})$, $P \in \mathrm{M}_{2g}(\mathbb{Z})$ such that $M = QP$ and $H(P) \leq C\mathcal{H}^\kappa$.

PROOF. Using elementary row operations from $\mathrm{GL}_{2g}(\mathbb{Z})$, we can write $M = M_1P_1$ with $M_1 \in \mathrm{GL}_{2g}(\mathbb{Z})$ and $P_1 \in \mathrm{M}_{2g}(\mathbb{Z})$ upper triangular. We can choose the row operations in the following way: With each row operation, we try to reduce the sum of the absolute values of the entries of the first column of the matrix. As soon as there is no elementary row operation from $\mathrm{GL}_{2g}(\mathbb{Z})$ that reduces this quantity, we stop. As the sum of the absolute values of the entries of the first column of M is some natural number B, we stop after at most B operations. The first column of the matrix thus obtained can then have at most one non-zero entry (else we could reduce the aforementioned quantity further) and it has in fact exactly one non-zero entry as $\det M \neq 0$. By permuting the rows, we can assume that the new matrix is of the form $\begin{pmatrix} * & * \\ 0 & M' \end{pmatrix}$. We proceed inductively with M'.

The (non-zero) diagonal entries of P_1 are then clearly bounded by $|\det M|$ and after more row operations we can assume that the entries above the diagonal entry d lie in the set $\{0, 1, \ldots, |d|-1\}$. So we can assume without loss of generality that $H(P_1)$ is bounded by $|\det M|$, which is of course polynomially bounded in $\mathcal{H}$. Then

$$\mathcal{H}' = H\left(M_1^t \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix} M_1\right)$$

is also polynomially bounded in $\mathcal{H}$, so it suffices to prove the lemma for M_1 and $\mathcal{H}'$ instead of M and $\mathcal{H}$.

The lemma is now a consequence of Orr's Lemma 4.3 in [124], which can be reformulated as asserting that there exists $P_2 \in \mathrm{GL}_{2g}(\mathbb{Z})$ of height bounded polynomially in $\mathcal{H}'$ such that $M_1P_2 \in \mathrm{Sp}_{2g}(\mathbb{Z})$. Note that a step is missing in the proof of Lemma 4.3 in [124] (we use the notation from there): In the construction of $e_2' = \sum_{i=1}^{2g} a_i e_i$ in that proof, one can assume without loss of generality that $a_1 = 0$. After that, one should first choose $e_3'', \ldots, e_{2g}''$ such that $\mathbb{Z}e_2' \oplus \mathbb{Z}e_3'' \oplus \cdots \oplus \mathbb{Z}e_{2g}'' = \mathbb{Z}e_2 \oplus \mathbb{Z}e_3 \oplus \cdots \oplus \mathbb{Z}e_{2g}$ and the coefficients of the e_i'' with respect to the e_j are polynomially bounded in N ($i = 3, \ldots, 2g$). This can be achieved inductively by following the proof of Theorem II.1 in [122] and, in each step of the induction, choosing ρ, σ such that $\max\{|\rho|, |\sigma|\} \leq \max\{|\delta_{n-1}|, |\alpha_n|\}$ (in the notation of [122]). Afterwards, one resumes the construction from the proof in [124], but with e_i'' in place of e_i ($i = 3, \ldots, 2g$). This yields a complete proof of Lemma 4.3 in [124] (with worse constants c, k than the ones given there). $\square$

Before we can state the next theorem, we have to define what a Siegel fundamental domain for the action of (a finite-index subgroup of) $\mathrm{Sp}_{2g}(\mathbb{Z})$ on $\mathbb{H}_g$ is. We give the definition that goes back to Siegel in [168], §2.

DEFINITION 3.3.2. *(1) A positive definite symmetric matrix*

$$M = (m_{ij})_{i,j=1,\dots,g} \in \mathrm{M}_g(\mathbb{R})$$

is called Minkowski-reduced if $v^t M v \geq m_{ii}$ *for all* $v^t = (v_1, \dots, v_g) \in \mathbb{Z}^g$ *with* $\gcd(v_i, \dots, v_g) = 1$ *and all* $i = 1, \dots, g$ *and* $m_{i,i+1} \geq 0$ *for all* $i = 1, \dots, g-1$.

(2) The Siegel fundamental domain for $\mathrm{Sp}_{2g}(\mathbb{Z})$ *or the Siegel fundamental domain is the set of* $\tau = (\tau_{ij})_{i,j=1,\dots,g} \in \mathbb{H}_g$ *such that* $\det(\mathrm{Im}(M[\tau])) \leq \det(\mathrm{Im}\,\tau)$ *for all* $M \in \mathrm{Sp}_{2g}(\mathbb{Z})$*,* $\mathrm{Im}\,\tau$ *is Minkowski-reduced, and* $|\mathrm{Re}\,\tau_{ij}| \leq \frac{1}{2}$ $(i, j = 1, \dots, g)$*.*

(3) If F *denotes the Siegel fundamental domain for* $\mathrm{Sp}_{2g}(\mathbb{Z})$*,* $G \subset \mathrm{Sp}_{2g}(\mathbb{Z})$ *is a subgroup of finite index, and* $g_1 = I_{2g}, g_2, \dots, g_n$ *is a system of representatives for its right cosets, then* $\bigcup_{j=1}^n g_j F$ *is called a Siegel fundamental domain for* G*.*

It is a classical fact that only for finitely many $M \in \mathrm{Sp}_{2g}(\mathbb{Z})$ there exists some τ in the Siegel fundamental domain with $M[\tau]$ also in the Siegel fundamental domain, and that every element of $\mathbb{H}_g$ can be brought into the Siegel fundamental domain by some element of $\mathrm{Sp}_{2g}(\mathbb{Z})$. The same facts then easily follow for every Siegel fundamental domain for some subgroup of $\mathrm{Sp}_{2g}(\mathbb{Z})$ of finite index. This is everything we will need to know about Siegel fundamental domains in this section.

PROPOSITION 3.3.3. *Let* A *and* B *be two abelian varieties of dimension* g*, defined over* $\mathbb{C}$ *and uniformized as* $\mathbb{C}^g/\Omega_A\mathbb{Z}^{2g}$ *and* $\mathbb{C}^g/\Omega_B\mathbb{Z}^{2g}$ *respectively, where* $\Omega_A = (T_A\ I_g)$ *and* $\Omega_B = (T_B\ I_g)$ *with* T_A*,* $T_B \in F$ *and* F *denotes a Siegel fundamental domain for a subgroup of* $\mathrm{Sp}_{2g}(\mathbb{Z})$ *of finite index. Let* $\mathcal{M}$ *and* $\mathcal{N}$ *be ample line bundles on* A *and* B *respectively which are associated to the Hermitian forms given by* $(\mathrm{Im}\,T_A)^{-1}$ *and* $(\mathrm{Im}\,T_B)^{-1}$ *respectively on* $\mathbb{C}^g$*.*

Let $\phi : A \to B$ *be an isogeny. Then there exist constants* C *and* κ*, depending only on* F*,* A*,* Ω_A*, and* $\mathcal{M}$ *(but not on* B *or* ϕ*), a natural number* $n \in \mathbb{N}$*, an automorphism* $\sigma : A \to A$*, and a matrix* $\Phi \in \mathrm{M}_{2g}(\mathbb{Z})$ *such that*

(i) $((\phi \circ \sigma)^*\mathcal{N})^{\otimes n} \otimes \mathcal{M}^{\otimes(-1)}$ *is ample and* $n \leq C(\deg\phi)^\kappa$*.*

(ii) Φ *is the rational representation of* $\phi \circ \sigma$ *with respect to the lattice bases given by* Ω_A *and* Ω_B *and* $H(\Phi) \leq C(\deg\phi)^\kappa$*.*

PROOF. Let $\phi_{\mathcal{M}}$ and $\phi_{\mathcal{N}}$ be the principal polarizations induced by $\mathcal{M}$ and $\mathcal{N}$ respectively. Consider $\psi = \phi_{\mathcal{M}}^{-1} \circ \hat{\phi} \circ \phi_{\mathcal{N}} \circ \phi \in \mathrm{End}(A)$. It is symmetric, i.e. $\psi' = \psi$, where $\psi' = \phi_{\mathcal{M}}^{-1} \circ \hat{\psi} \circ \phi_{\mathcal{M}}$ denotes the Rosati involution. It is also totally positive (or positive definite in the terminology of [124]) by Theorem 5.2.4 in [19] since $\hat{\phi} \circ \phi_{\mathcal{N}} \circ \phi = \phi_{\phi^*\mathcal{N}}$ is a polarization of A.

Therefore, we can apply Orr's Proposition 4.2 from [124] and deduce that there is a constant c, depending only on A and Ω_A, and $\sigma \in \mathrm{Aut}(A)$ such that the rational representation of $\sigma' \circ \psi \circ \sigma$ with respect to the lattice given by Ω_A has height bounded by $c(\deg\phi)^2$ (we choose $(\mathrm{End}\,A, ')$ as $(R, \dagger)$ and the rational representation with respect to the lattice given by Ω_A as ρ). We have $\sigma' \circ \psi \circ \sigma = \phi_{\mathcal{M}}^{-1} \circ \widehat{(\phi \circ \sigma)} \circ \phi_{\mathcal{N}} \circ (\phi \circ \sigma)$, so we can replace ϕ by $\phi \circ \sigma$ and ψ by $\sigma' \circ \psi \circ \sigma$ and verify (i) and (ii) for this new ϕ (and $\sigma = \mathrm{id}$), where $\Phi \in \mathrm{M}_{2g}(\mathbb{Z})$ is the rational representation of ϕ with respect to the lattice bases given by Ω_A and Ω_B. We have $|\det \Phi| = \deg \phi \neq 0$.

Let $H_{\mathcal{M}}$ and $H_{\mathcal{N}}$ be the Hermitian forms on $\mathbb{C}^g$ associated to $\mathcal{M}$ and $\mathcal{N}$ respectively and let A' and B' be the matrices in $\mathrm{M}_{2g}(\mathbb{R})$ that represent the symmetric positive definite forms $\operatorname{Re} H_{\mathcal{M}}$ and $\operatorname{Re} H_{\mathcal{N}}$ with respect to the lattice bases given by Ω_A and Ω_B respectively. Let $M_1 \in \mathrm{M}_{2g}(\mathbb{Z})$ be the rational representation of ψ with respect to the lattice basis given by Ω_A. Now ψ satisfies $\phi_{\mathcal{M}} \circ \psi = \hat{\phi} \circ \phi_{\mathcal{N}} \circ \phi$. By taking the analytic representations of both sides, where the dual abelian varieties are canonically uniformized as quotients of the vector space of $\mathbb{C}$-antilinear maps from $\mathbb{C}^g$ to $\mathbb{C}$, it follows (with Lemma 2.4.5 from [19]) that

$$H_{\mathcal{M}}(\psi(v), w) = H_{\mathcal{N}}(\phi(v), \phi(w))$$

for all $v, w \in \mathbb{C}^g$, where we use ϕ and ψ also for the linear maps from $\mathbb{C}^g$ to $\mathbb{C}^g$ corresponding to the analytic representations of ϕ and ψ with respect to the given uniformization. By taking real parts and passing to rational representations, we deduce that

$$(M_1 v)^t A' w = v^t \Phi^t B' \Phi w$$

for all $v, w \in \mathbb{R}^{2g}$ and it follows that $M_1^t = \Phi^t B' \Phi (A')^{-1}$ and therefore $M_1 = (A')^{-1} \Phi^t B' \Phi$.

Let $H_{\phi^*\mathcal{N}}$ be the Hermitian form associated to $\phi^*\mathcal{N}$. The ampleness of $(\phi^*\mathcal{N})^{\otimes n} \otimes \mathcal{M}^{\otimes(-1)}$ is equivalent to the positive definiteness of its Hermitian form $H_n = nH_{\phi^*\mathcal{N}} - H_{\mathcal{M}}$ and this is equivalent to the positive definiteness of the symmetric bilinear form $\operatorname{Re} H_n$. One computes that $\operatorname{Re} H_n$ is represented by $M_2 = n\Phi^t B' \Phi - A'$ with respect to the lattice given by Ω_A. Let $v \in \mathbb{R}^{2g}$ be an arbitrary non-zero vector and $M_3 = \Phi^t B' \Phi$. Then we have

$$v^t M_2 v = n v^t M_3 v - v^t M_3 (M_1^{-1} v),$$

and using the Cauchy-Schwarz inequality for the scalar product given by M_3 we obtain

$$v^t M_2 v \geq \sqrt{v^t M_3 v} \left(n \sqrt{v^t M_3 v} - \sqrt{(M_1^{-1} v)^t M_3 (M_1^{-1} v)} \right).$$

In order to make this quantity positive, n must be bigger than the operator norm of M_1^{-1} with respect to the scalar product given by M_3, i.e.

$$n > \sqrt{(M_1^{-1} v_0)^t M_3 (M_1^{-1} v_0)}$$

for every $v_0 \in \mathbb{R}^{2g}$ with $v_0^t M_3 v_0 = 1$.

We know from Orr's proposition that all coefficients of M_1 are bounded by $c(\deg \phi)^2$. Therefore, we can bound the coefficients of both $M_3 = A' M_1$ and $M_3^{-1} = M_1^{-1} (A')^{-1}$ by some power of $\deg \phi$ times a constant, where the constant depends only on A, Ω_A, and $\mathcal{M}$ (note that $|\det M_1| = (\deg \phi)^2 \geq 1$, so we have a similar bound for the coefficients of M_1^{-1} as for the coefficients of M_1).

Since B' and hence M_3 is symmetric and positive definite, there is a matrix $\tilde{M_3} \in \mathrm{GL}_{2g}(\mathbb{R})$ such that $M_3 = \tilde{M_3}^t \tilde{M_3}$. We can then write $M_3^{-1} = \tilde{M_3}^{-1} \tilde{M_3}^{-t}$, so the coefficients of $\tilde{M_3}$ and $\tilde{M_3}^{-1}$ must be similarly bounded.

When $(\tilde{M}_3 v_0)^t \tilde{M}_3 v_0 = 1$, the coordinates of $\tilde{M}_3 v_0$ are at most 1 in absolute value. Hence, those of $v_0 = \tilde{M}_3^{-1}(\tilde{M}_3 v_0)$ are also bounded by some power of $\deg \phi$ times a constant which depends only on A, Ω_A, and $\mathcal{M}$. Finally we fix n to be the largest integer with

$$n \leq \sqrt{(M_1^{-1} v_0)^t M_3 (M_1^{-1} v_0)} + 1$$

and obtain a bound of the desired form. This proves (i).

For (ii), we have $\Phi^t[T_B] = T_A$ for the partial action of $GL_{2g}(\mathbb{Q})$ on $\mathbb{H}_g$ that restricts to the usual action of $\mathrm{Sp}_{2g}(\mathbb{Z})$.

By Lemma 3.3.1, we can write $\Phi = QP$, where $Q \in \mathrm{Sp}_{2g}(\mathbb{Z})$ and $P \in \mathrm{M}_{2g}(\mathbb{Z})$ with $H(P)$ bounded polynomially in $H\left(\Phi^t \left(\begin{smallmatrix} 0 & I_g \\ -I_g & 0 \end{smallmatrix}\right) \Phi\right)$. But now $\Phi^t \left(\begin{smallmatrix} 0 & I_g \\ -I_g & 0 \end{smallmatrix}\right) \Phi$ represents the imaginary part of the Hermitian form $H_{\phi^*\mathcal{N}}$ with respect to the lattice basis given by Ω_A (here we use that the lattice basis associated to Ω_B is symplectic with respect to $H_{\mathcal{N}}$). We have

$$|\operatorname{Im} H_{\phi^*\mathcal{N}}(v,w)|^2 \leq |H_{\phi^*\mathcal{N}}(v,w)|^2 \leq H_{\phi^*\mathcal{N}}(v,v) H_{\phi^*\mathcal{N}}(w,w)$$

by Cauchy-Schwarz, where $v, w \in \mathbb{C}^g$.

Furthermore, we know that

$$H_{\phi^*\mathcal{N}}(v,v) H_{\phi^*\mathcal{N}}(w,w) = \operatorname{Re} H_{\phi^*\mathcal{N}}(v,v) \operatorname{Re} H_{\phi^*\mathcal{N}}(w,w).$$

But $\operatorname{Re} H_{\phi^*\mathcal{N}}$ is represented by $M_3 = \Phi^t B' \Phi$ and we have already bounded the coefficients of that matrix. So the coefficients of $\Phi^t \left(\begin{smallmatrix} 0 & I_g \\ -I_g & 0 \end{smallmatrix}\right) \Phi$ are also bounded polynomially in $\deg \phi$, and as they are integers, their height is similarly bounded.

This means we have written $\Phi = QP$, where $Q \in \mathrm{Sp}_{2g}(\mathbb{Z})$ and $P \in \mathrm{M}_{2g}(\mathbb{Z})$ with $H(P)$ polynomially bounded in $\deg \phi$. Furthermore, $Q^t[T_B]$ is an element of $\mathbb{H}_g$. There is $R \in \mathrm{Sp}_{2g}(\mathbb{Z})$ such that $(RQ^t)[T_B]$ lies again in the Siegel fundamental domain. By [134], Lemma 3.2, the height of R is polynomially bounded in terms of the maximum of the absolute values of the coefficients of $Q^t[T_B]$ together with 1 and $(\det \operatorname{Im} Q^t[T_B])^{-1}$. Note that such a bound holds for the Siegel fundamental domain as defined here although in [134] Siegel's definition from [169] is used, which demands that $(\operatorname{Im} \tau)^{-1}$ instead of $\operatorname{Im} \tau$ is Minkowski-reduced, since by Lemma 3.3 in [134] and Lemma 3.1(3) in [135] one can switch between the two fundamental domains in a (polynomially) controlled way.

In order to bound the absolute values of the coefficients of $Q^t[T_B]$ as well as $(\det \operatorname{Im} Q^t[T_B])^{-1}$, we consider the matrix $M_4 = Q^t B' Q = P^{-t} M_3 P^{-1}$. Recall that $\det P = \det \Phi \neq 0$. As we have a bound on the coefficients of M_3 and on $H(P)$, we deduce a similar bound for the coefficients of M_4. If we write $Q = \left(\begin{smallmatrix} Q_1^t & Q_3^t \\ Q_2^t & Q_4^t \end{smallmatrix}\right)$, then we see that M_4 represents the real part of $H_{\mathcal{N}}$ with respect to the lattice basis given by the columns of $\left(\begin{smallmatrix} T_B Q_1^t + Q_2^t & T_B Q_3^t + Q_4^t \end{smallmatrix}\right)$.

In order to compute M_4, it is useful to choose the basis given by the columns of $T_B Q_3^t + Q_4^t$ for $\mathbb{C}^g$. That this matrix has non-zero determinant (and hence its columns form a basis) follows from the proof that $\mathrm{Sp}_{2g}(\mathbb{Z})$ acts on $\mathbb{H}_g$ by $(U, \tau) \mapsto U[\tau]$.

With respect to this new basis of $\mathbb{C}^g$, the lattice basis given by the columns of $\left(\begin{smallmatrix} T_B Q_1^t + Q_2^t & T_B Q_3^t + Q_4^t \end{smallmatrix}\right)$ is given by the matrix $\left(\begin{smallmatrix} Q^t[T_B] & I_g \end{smallmatrix}\right)$. Furthermore, the Hermitian form $H_{\mathcal{N}}$ is given by $(Q_3 T_B + Q_4)(\operatorname{Im} T_B)^{-1}(\overline{T_B} Q_3^t + Q_4^t) = (\operatorname{Im} Q^t[T_B])^{-1}$ with respect to this new basis of $\mathbb{C}^g$ (see the calculation in [19], p. 214).

With this new basis for $\mathbb{C}^g$, it is easy to compute

$$M_4 = \begin{pmatrix} M_5 & (\operatorname{Re} Q^t[T_B])(\operatorname{Im} Q^t[T_B])^{-1} \\ (\operatorname{Im} Q^t[T_B])^{-1}(\operatorname{Re} Q^t[T_B]) & (\operatorname{Im} Q^t[T_B])^{-1} \end{pmatrix},$$

where

$$M_5 = (\operatorname{Re} Q^t[T_B])(\operatorname{Im} Q^t[T_B])^{-1}(\operatorname{Re} Q^t[T_B]) + \operatorname{Im} Q^t[T_B].$$

Here, we used that $Q^t[T_B]$ and hence both its real and imaginary part are symmetric.

Now our bound on the coefficients of M_4 yields first an upper bound on the coefficients of M_5 and on $(\det \operatorname{Im} Q^t[T_B])^{-1}$. Next, we deduce $\det \operatorname{Im} Q^t[T_B] \leq \det M_5$ from Minkowski's determinant inequality (see [94], Chapter II, Theorem 4.1.8) since both $\operatorname{Im} Q^t[T_B]$ and $M_5 - \operatorname{Im} Q^t[T_B]$ are symmetric and positive semidefinite. From this follows an upper bound for $\det \operatorname{Im} Q^t[T_B]$. Together with our bound on the coefficients of M_4, this readily gives a bound for the coefficients of $\operatorname{Im} Q^t[T_B]$ and $\operatorname{Re} Q^t[T_B]$ and thereby a bound for the coefficients of $Q^t[T_B]$ in absolute value. Thus, we can apply Lemma 3.2 from [134] to bound $H(R)$ in the required way.

We note that T_B lies in the Siegel fundamental domain of a finite-index subgroup of $\mathrm{Sp}_{2g}(\mathbb{Z})$ and $(RQ^t)[T_B]$ lies in the Siegel fundamental domain of $\mathrm{Sp}_{2g}(\mathbb{Z})$ itself. Therefore, RQ^t has to lie in a certain finite set which depends only on F and g. Thus, we obtain a similar bound for $H(Q) = H(R^{-1}RQ^t)$ and thereby for $H(\Phi) = H(QP)$ since we have already bounded $H(P)$. □

In order to state the next corollary, we introduce the following notation that will also be used in the following sections: We write $f \preceq g$ for (positive) quantities f and g, if there exist constants $c > 0$ and $\kappa > 0$, depending on K, A_0, L_0, τ_0, Γ, l, $\mathcal{A}$, $\mathcal{L}$, S, $\mathcal{C}$, and the choice of a Siegel fundamental domain for $G(l, 2l)$ such that

$$f \leq c \max\{1, g\}^{\kappa}. \tag{3.3.1}$$

The choice of a Siegel fundamental domain for $G(l, 2l)$ will be made implicitly in Proposition 3.6.1.

Corollary 3.3.4. *Let $s \in S$ such that A_0 and $\mathcal{A}_s$ are isogenous. Choose τ in a Siegel fundamental domain for the action of $G(l, 2l)$ on $\mathbb{H}_g$ such that $\iota(\tau) = s$ with ι as in Proposition 3.2.1. Then there exist an isogeny $\phi_s : A_0 \to \mathcal{A}_s$ of minimal degree (as defined before (3.2.2)), a natural number $M \in \mathbb{N}$, and a matrix $\Phi \in \mathrm{M}_{2g}(\mathbb{Z})$ such that*

(i) $(\phi_s^* \mathcal{L}_s)^{\otimes M} \otimes L_0^{\otimes(-1)}$ *is ample and* $M \preceq \deg \phi_s$.

(ii) Φ *is the rational representation of* ϕ_s *with respect to the uniformizations* $\exp_0$ *and* $\exp(\tau, \cdot)$ *and the lattice bases given by* Ω_τ *and* Ω_{τ_0}, *where* $\exp$, $\exp_0$, Ω_τ, *and* Ω_{τ_0} *are defined as in Section 3.2. It satisfies* $H(\Phi) \preceq \deg \phi_s$.

Proof. Let ϕ be any isogeny of minimal degree between A_0 and $\mathcal{A}_s$. We apply Proposition 3.3.3 to ϕ with $A = A_0$, $B = \mathcal{A}_s$, $T_A = \tau_0$, $T_B = \tau$, $\mathcal{M} = L_0$, and $\mathcal{N} = \mathcal{L}'_s$, where $\mathcal{L}_s = (\mathcal{L}'_s)^{\otimes l}$ by Proposition 3.2.1(vi). Putting $\phi_s = \phi \circ \sigma$ yields what we want: Since σ is an automorphism, we have $\deg \phi_s = \deg \phi$, so ϕ_s is of minimal degree. As $(\phi_s^* \mathcal{L}'_s)^{\otimes n} \otimes L_0^{\otimes(-1)}$ is ample, so is $(\phi_s^* \mathcal{L}'_s)^{\otimes ln} \otimes L_0^{\otimes(-l)} \otimes L_0^{\otimes(l-1)} = (\phi_s^* \mathcal{L}_s)^{\otimes n} \otimes L_0^{\otimes(-1)}$ and thus we may take $M = n$. Note that $G(l, 2l)$ has finite index in $\mathrm{Sp}_{2g}(\mathbb{Z})$ and that τ_0 was already chosen in the Siegel fundamental domain for

$\mathrm{Sp}_{2g}(\mathbb{Z})$. The implicit constants depend only on A_0, L_0, τ_0, and the chosen Siegel fundamental domain, but are independent of s and τ. $\square$

Finally, we record a lemma due to Rémond that allows us to bound the height of a basis of the lattice corresponding to an abelian subvariety of A_0 in terms of the degree of the abelian subvariety.

LEMMA 3.3.5. *Let B_0 be an abelian subvariety of A_0 of codimension k and denote by $\deg B_0$ its degree with respect to the ample line bundle L_0. Under the identification of $\mathbb{R}^{2g}$ with $\mathbb{C}^g$ given by $u \mapsto \Omega_{\tau_0} u$, there exists a matrix $H \in \mathrm{M}_{2g\times 2(g-k)}(\mathbb{Z})$ such that $\exp_0^{-1}(B_0(\mathbb{C})) = \{Hy + z; y \in \mathbb{R}^{2(g-k)}, z \in \mathbb{Z}^{2g}\}$, $\Omega_{\tau_0} H$ has rank equal to $g-k$, and $\|H\| \preceq \deg B_0$. Here, $\exp_0$ and Ω_{τ_0} are defined as in Section 3.2.*

PROOF. We follow Rémond's construction in Section 4 of [149]. We obtain a basis $w_i = \sum_{j=1}^{2g} \lambda_j^{(i)} v_j$ $(i = 1, \ldots, 2(g-k))$ of the connected component of $\exp_0^{-1}(B_0(\mathbb{C}))$ containing 0 under the given identification of $\mathbb{R}^{2g}$ and $\mathbb{C}^g$. Here, $v_1, \ldots, v_{2g}$ is a suitable basis of $\mathbb{Z}^{2g}$ that is chosen depending on L_0, but independently of B_0, and the $\lambda_j^{(i)}$ are integers.

By an inequality on p. 531 of [149], we have

$$\|w_i\|\|w_{2(g-k)+1-i}\| \preceq \deg B_0 \quad (i = 1, \ldots, 2(g-k)),$$

where $\|\cdot\|$ is a Euclidean norm on $\mathbb{R}^{2g}$ induced by L_0. This norm is bounded from below on $\mathbb{Z}^{2g}\backslash\{0\}$ by a positive constant that does not depend on B_0, which implies that

$$\|w_i\| \preceq \deg B_0 \quad (i = 1, \ldots, 2(g-k)).$$

Since all norms on finite-dimensional real vector spaces are equivalent and $\|\cdot\|$ does not depend on B_0, it follows that $|\lambda_j^{(i)}| \preceq \deg B_0$ $(i = 1, \ldots, 2(g-k)$, $j = 1, \ldots, 2g)$. We deduce that the coordinates of $w_1, \ldots, w_{2(g-k)}$ with respect to the basis $v_1, \ldots, v_{2g}$ of $\mathbb{Z}^{2g}$ (which is not necessarily the standard one) are bounded. However, this basis is chosen independently of B_0, and so we obtain a comparable bound for the coordinates with respect to the standard basis.

We now take as H the matrix with columns $w_1, \ldots, w_{2(g-k)}$. The columns of the matrix $\Omega_{\tau_0} H$ span the connected component of $\exp_0^{-1}(B_0(\mathbb{C}))$ containing 0 seen as a $(g-k)$-dimensional vector subspace of $\mathbb{C}^g$ and so this matrix has rank equal to $g-k$. $\square$

3.4. Galois orbit bounds

In this section, we show that virtually all occurring important quantities can be bounded polynomially in terms of $[K(p) : K]$, where p is a point in $\mathcal{A}_\Gamma^{[k]} \cap \mathcal{C}$ (reversing the direction of the inequalities leads to lower bounds for $[K(p) : K]$ in terms of these other quantities – hence the title "Galois orbit bounds"). We will need two lemmata before we can prove the crucial Proposition 3.4.3. From now on, we will always take the isogeny given by Corollary 3.3.4 as ϕ_s. There might be some ambiguity in the choice of τ if it lies on the boundary of the Siegel fundamental domain for $G(l, 2l)$, but this ambiguity does not change the construction in Proposition 3.3.3 – which only depends on the principal polarization induced by $\mathcal{L}'_s$ and the data associated to A_0 – and hence has no influence on ϕ_s. Likewise, the implicit constants in the estimates are the same for any choice of τ in the Siegel fundamental domain.

LEMMA 3.4.1. *Let $s \in S$ be such that $\mathcal{A}_s$ and A_0 are isogenous. Then there are constants c_1 and c_2, depending on K and A_0, but independent of s such that*

$$h_{\overline{S}}(s) \leq c_1 \log[K(s) : K] + c_2.$$

PROOF. We will use $c_1, c_2, \ldots$ for constants depending on K and A_0, but independent of s. We will denote the stable Faltings height of $\mathcal{A}_s$ as defined in [42] by $h_F(\mathcal{A}_s)$.

By Faltings' Lemma 5 in [42], we have

$$h_F(\mathcal{A}_s) \leq h_F(A_0) + \frac{\log \deg \phi_s}{2}. \tag{3.4.1}$$

By an inequality of Bost-David (Pazuki's Corollary 1.3 (1) in [126]), we know that

$$\left| h_{\overline{S}}(s) - \frac{1}{2} h_F(\mathcal{A}_s) \right| \leq c_3 \log(\max\{h_{\overline{S}}(s), 1\}) + c_4$$

for some constants c_3 and c_4, depending only on g and l. Our choice of embedding of $A_{g,(2l,l)}$ and $\mathfrak{A}_{g,(2l,l)}$ into projective space through the use of Theta functions means that our $h_{\overline{S}}(s)$ differs from the Theta height of $\mathcal{A}_s$ in Pazuki's work with $l = r^2$ only by an amount that is bounded independently of s: Pazuki uses another norm at the infinite places for the definition of his height and he uses another coordinate system as he notes after his Definition 2.6, but by [75], p. 171, this coordinate system is related to ours by an invertible linear transformation with algebraic coefficients.

We deduce that

$$h_{\overline{S}}(s) \leq c_5 \max\{h_F(\mathcal{A}_s), 1\}. \tag{3.4.2}$$

Combining (3.2.2), (3.4.1), and (3.4.2), we obtain that

$$h_{\overline{S}}(s) \leq c_1 \log[K(s) : K] + c_2$$

for some constants c_1 and c_2. □

LEMMA 3.4.2. *Let $p \in \mathcal{C}$ with $s = \pi(p) \in S$ and suppose that $\mathcal{C}$ is not contained in $\mathcal{A}_s$. Then we have $\widehat{h}_s(p) \preceq h_{\overline{S}}(s)$.*

Our proof even yields a bound that is linear in $h_{\overline{S}}(s)$, but a polynomial bound will suffice for our purposes. We note that it is crucial for this lemma that $\mathcal{C}$ is a curve and not a subvariety of $\mathcal{A}$ of higher dimension. Indeed, the main obstacle that one encounters attempting to generalize Theorem 3.1.3 to higher-dimensional subvarieties $\mathcal{V} \subset \mathcal{A}$ which dominate the base is the lack of such a height bound for (a large enough subset of) the points in $\mathcal{A}_\Gamma \cap \mathcal{V}$.

PROOF. We use $c_6, \ldots$ for constants that depend only on $\mathcal{A}$ and $\mathcal{C}$. Let for the moment $s \in S$ and $p \in \mathcal{A}_s$ be arbitrary. We will first bound $\widehat{h}_s(p)$ in terms of $h_s(p)$ and $h_{\overline{S}}(s)$. It would be possible to use Silverman's Theorem A from [171] for this; there is however the problem that $\overline{\mathfrak{A}_{g,(2l,l)}}$ and $\overline{A_{g,(2l,l)}}$ are usually not smooth, so one would either need to construct a more sophisticated (i.e. smooth) compactification of the universal family (this was achieved by Pink in his dissertation [139]) or adapt Silverman's proof by using Cartier instead of Weil divisors.

Another, more elementary way is to use Lemma 3.4 from [101]. It is shown in that lemma that there exists a family of polynomials $P_{i,j}$ ($i = 0, \dots, l^g - 1$, $j = 1, \dots, J$) in the projective coordinates of $s \in S$ and $p \in \mathcal{A}_s \subset \mathbb{P}^{l^g-1}$ with the following properties: Every $P_{i,j}$ is a polynomial with integer coefficients, homogeneous of degree $2(l^{8g}-1)$ in the coordinates of s and homogeneous of degree 4 in the coordinates of p. For every $s \in S$ and $p \in \mathcal{A}_s$ and every $j \in \{1, \dots, J\}$, the $P_{i,j}(s,p)$ ($i = 0, \dots, l^g - 1$) are either all zero or they are the projective coordinates of $2p$ in $\mathcal{A}_s \subset \mathbb{P}^{l^g-1}$ (by abuse of notation, $P_{i,j}(s,p)$ denotes $P_{i,j}$ evaluated at the projective coordinates of s and p). Furthermore, there exists $j \in \{1, \dots, J\}$, depending on s and p, such that not all $P_{i,j}(s,p)$ ($i = 0, \dots, l^g - 1$) are zero.

Fixing $j \in \{1, \dots, J\}$ and following the proof of Theorem B.2.5(a) in [73] (which amounts to the triangle inequality), we get a bound of the form

$$h_s(2p) \leq 4h_s(p) + 2(l^{8g} - 1)h_{\overline{S}}(s) + c_6,$$

where c_6 depends only on l, g, and the (integral) coefficients of the $P_{i,j}$, but is independent of s and p. The bound is valid for those s and p where not all $P_{i,j}(s,p)$ ($i = 0, \dots, l^g - 1$) are zero. After reiterating the process for every $j \in \{1, \dots, J\}$ and adjusting the constants if necessary, we can assume that the inequality holds for all $s \in S$ and $p \in \mathcal{A}_s$. We then obtain easily from $\widehat{h}_s(p) = \lim_{n\to\infty} \frac{h_s(2^n p)}{4^n}$ that

$$\widehat{h}_s(p) \leq h_s(p) + \frac{2(l^{8g} - 1)h_{\overline{S}}(s) + c_6}{3},$$

where we used that $\sum_{n=1}^{\infty} 4^{-n} = \frac{1}{3}$.

Let now p be a point of $\mathcal{C}$ as in the lemma. In view of the above inequality, it suffices to show that $h_s(p) \preceq h_{\overline{S}}(s)$. Since $\mathcal{C}$ is irreducible and not contained in $\mathcal{A}_s$, the morphism $\pi|_{\overline{\mathcal{C}}} : \overline{\mathcal{C}} \to \overline{S}$ is quasi-finite. It is also proper, hence finite. Therefore, the pull-back $\pi^* L_{\overline{S}}$ of the ample line bundle $L_{\overline{S}}$ is also ample.

On the other hand, the closed immersion $\iota : \overline{\mathcal{C}} \hookrightarrow \overline{\mathfrak{A}_{g,(2l,l)}}$ yields a very ample line bundle $\iota^*\mathcal{L}$ on $\overline{\mathcal{C}}$. It follows from the ampleness of $\pi^* L_{\overline{S}}$ that there exists some natural number $N \in \mathbb{N}$ such that $\pi^* L_{\overline{S}}^{\otimes N} \otimes \iota^*\mathcal{L}^{\otimes(-1)}$ is ample.

If we choose associated heights $h_{\overline{\mathcal{C}},\iota^*\mathcal{L}}$ and $h_{\overline{\mathcal{C}},\pi^* L_{\overline{S}}}$, it now follows from fundamental properties of the Weil height that

$$h_{\overline{\mathcal{C}},\iota^*\mathcal{L}}(p) \leq N h_{\overline{\mathcal{C}},\pi^* L_{\overline{S}}}(p) + c_7$$

and then by functoriality that

$$h_s(p) \leq N h_{\overline{S}}(s) + c_8,$$

whence the lemma follows. □

The next proposition bounds all important quantities in terms of $[K(p) : K]$ alone, where p is some point in $\mathcal{A}_\Gamma^{[k]} \cap \mathcal{C}$.

PROPOSITION 3.4.3. *Let $s \in S$ be such that $\mathcal{A}_s$ and A_0 are isogenous and $p \in \mathcal{C} \cap \phi_s(\Gamma + B_0)$ for some abelian subvariety B_0 of A_0. Suppose that $\pi(\mathcal{C}) = S$. Then there exist $\gamma \in \Gamma$, an abelian subvariety $B_1 \subset B_0$, and $b \in B_1$ with the following properties: If we choose $N \in \mathbb{N}$ minimal with $N\gamma = \sum_{i=1}^{r} a_i \gamma_i \in \mathbb{Z}\gamma_1 + \dots + \mathbb{Z}\gamma_r$ and if $\deg B_1$ denotes the degree of B_1 with respect to the ample line bundle L_0, then we have $p = \phi_s(\gamma + b)$ and*

(i) $\deg \phi_s \preceq [K(p) : K]$,
(ii) $\deg B_1 \preceq [K(p) : K]$,
(iii) $\max\{|a_1|, \ldots, |a_r|, N\} \preceq [K(p) : K]$.

PROOF. Part (i) is just a restatement of (3.2.2), where we take into account that $[K(p) : K] \geq [K(s) : K]$. We have $p = \phi_s(q)$ for some $q \in \Gamma + B_0$. It follows from Corollary 3.3.4 that there exists $M \in \mathbb{N}$ such that $(\phi_s^* \mathcal{L}_s)^{\otimes M} \otimes L_0^{\otimes(-1)}$ is ample and $M \preceq \deg \phi_s \preceq [K(p) : K]$. Therefore

$$\widehat{h}_{A_0}(q) \leq M \widehat{h}_{\phi_s^*(\mathcal{L}_s)}(q) = M \widehat{h}_s(\phi_s(q)) = M \widehat{h}_s(p),$$

which implies together with Lemma 3.4.1 and Lemma 3.4.2 that $\widehat{h}_{A_0}(q) \preceq [K(p) : K]$.

We note that q is defined over a field extension of $K(p)$ of degree at most $\eta(g) \deg \phi_s$ for a certain function $\eta : \mathbb{N} \to \mathbb{N}$ since ϕ_s is defined over a field extension of $K(s) \subset K(p)$ of degree at most $\eta(g)$ by Rémond's Théorème 1.2 in [159] and q has degree at most $\deg \phi_s$ over the compositum of $K(p)$ and the field of definition of ϕ_s since all its Galois conjugates over that field lie in $\phi_s^{-1}(p)$ and this fiber has $\deg \phi_s$ elements. Here, Rémond has obtained the best possible η, while the fact that the bound depends only on g goes back to Silverberg in [170] and Masser-Wüstholz in [101], Lemma 2.1. Hence, we have $[K(q) : K] \preceq [K(p) : K]$.

Consider the point $\tilde{q} = (q, \gamma_1, \ldots, \gamma_r) \in A_0^{r+1}$. Let B be the smallest abelian subvariety of A_0^{r+1} such that a multiple $\mu\tilde{q}$ of $\tilde{q}$ lies inside B ($\mu \in \mathbb{N}$). By Proposition 9.1 in [70], we have $\deg B \preceq \max\{\widehat{h}_{A_0}(q), [K(q) : K]\} \preceq [K(p) : K]$. Here, $\deg B$ denotes the degree of B with respect to the line bundle $\pi_1^* L_0 \otimes \cdots \otimes \pi_{r+1}^* L_0$, where $\pi_i : A_0^{r+1} \to A_0$ is the projection to the i-th factor ($i = 1, \ldots, r+1$).

Let Ω be a finite set of abelian varieties over $\bar{\mathbb{Q}}$ such that every quotient A_0^{r+1}/H for some abelian subvariety H of A_0^{r+1} is isogenous over $\bar{\mathbb{Q}}$ to some element of Ω. For each $A' \in \Omega$ we can fix some norm $\|\cdot\|_{A'}$ on $\operatorname{Hom}(A_0^{r+1}, A') \otimes \mathbb{R}$ and a symmetric ample line bundle on A' to obtain a canonical height $\widehat{h}_{A'}$ on A'. After passing to a finite field extension, we can assume that all $A' \in \Omega$, all these line bundles as well as all elements of $\operatorname{Hom}(A_0^{r+1}, A')$ for all $A' \in \Omega$ are defined over K.

Going through the proof of Proposition 9.1 in [70], we see that B is obtained as the irreducible component of $\ker \alpha$ containing the neutral element for a surjective homomorphism $\alpha : A_0^{r+1} \to A$ for some $A \in \Omega$. If we write $\|\cdot\| = \|\cdot\|_A$, then we even obtain from Lemma 9.5 in [70] a surjective homomorphism $\alpha : A_0^{r+1} \to A$ such that B is the irreducible component of $\ker \alpha$ containing the neutral element and $\|\alpha\| \preceq [K(p) : K]$.

We have a projection morphism $\psi : B \to A_0^r$ given by omitting the first coordinate. We let $B' = \psi(B) \subset A_0^r$ and let B_2 be the connected component of $\ker \psi = B \cap (A_0 \times \{0\}^r) \subset B$ containing the neutral element. Since $q \in \Gamma + B_0$, it follows that $B_2 \subset B_0 \times \{0\}^r$. By Poincaré's reducibility theorem, there exists an abelian subvariety $B_3 \subset B$ such that the restriction of the natural addition morphism $B_2 \times B_3 \to B$ is an isogeny. It follows that $\psi|_{B_3} : B_3 \to B'$ must be an isogeny. As usual, there exists an isogeny $\chi : B' \to B_3$ such that $\chi \circ \psi|_{B_3}$ is multiplication by $\deg \psi|_{B_3}$ on B_3.

Since $\psi|_{B_3} : B_3 \to B'$ is surjective, we can choose $u \in B_3$ such that $\psi(u) = \mu(\gamma_1, \ldots, \gamma_r)$. Applying Poincaré's reducibility theorem again, we find an abelian subvariety $B'' \subset A_0^r$ such that the restriction of the natural addition morphism $B' \times B'' \to A_0^r$ is an isogeny. Again, we get an isogeny $\rho : A_0^r \to B' \times B''$ in the other direction such that their composition is multiplication by a scalar. By projecting to the first coordinate, we obtain $\rho' : A_0^r \to B'$. Let $w \in A_0^r$ with $\rho(w) = (\mu(\gamma_1, \ldots, \gamma_r), 0)$. It follows that $\mu(\gamma_1, \ldots, \gamma_r)$ is some multiple of w and hence $w \in \Gamma^r$. We have $(\deg \psi|_{B_3})u = \chi(\psi(u)) = \chi(\mu(\gamma_1, \ldots, \gamma_r)) = (\chi \circ \rho')(w)$. As $\chi \circ \rho' : A_0^r \to B_3 \hookrightarrow A_0^{r+1}$, Γ is stable under $\mathrm{End}(A_0)$, and $w \in \Gamma^r$, we deduce that $u \in \Gamma^{r+1}$.

It follows from $\psi(u) = \mu(\gamma_1, \ldots, \gamma_r)$ that $\mu(q, \gamma_1, \ldots, \gamma_r) \in u + \ker \psi \subset u + (A_0)_{\mathrm{tors}}^{r+1} + B_2$, and by considering only the first coordinate we see that $\mu q \in \pi_1(u) + (A_0)_{\mathrm{tors}} + \pi_1(B_2) \subset \Gamma + \pi_1(B_2)$ and hence $q \in \Gamma + \pi_1(B_2)$. Now, $B_1 = \pi_1(B_2)$ is an abelian subvariety of A_0 of degree $\deg B_1 = \deg B_2$ with respect to L_0. Since B_2 is an irreducible component of $B \cap (A_0 \times \{0\}^r)$, we know that $\deg B_2 \preceq \deg B$ by Proposition 3.1 in [158]. We also know that $B_2 = B_1 \times \{0\}^r$ and so $B_1 \subset B_0$ since $B_2 \subset B_0 \times \{0\}^r$. This proves (ii).

Since $\mathrm{Hom}(A_0^{r+1}, A)$ is a finitely generated $\mathbb{Z}$-module and the height is quadratic, there exists a constant c_0, depending only on the two abelian varieties and the choices of symmetric ample line bundles as well as the choice of the norm, such that $\widehat{h}_A(\alpha'(x)) \leq c_0 \|\alpha'\|^2 \sum_{i=1}^{r+1} \widehat{h}_{A_0}(x_i)$ for all $\alpha' \in \mathrm{Hom}(A_0^{r+1}, A)$ and all $x = (x_1, \ldots, x_{r+1}) \in A_0^{r+1}$. In particular, this bound holds for our α as chosen above.

We apply Rémond's Lemme 6.1 from [156] to choose $\gamma' \in \Gamma$ and $b' \in B_1$ such that $q = \gamma' + b'$ and $\widehat{h}_{A_0}(\gamma') \preceq \widehat{h}_{A_0}(q) \preceq [K(p) : K]$. Note that we have assumed $\Gamma = \Gamma_{sat}$ in Rémond's notation and that Rémond's Lemme also holds for $\epsilon = 0$ as is the case here. Suppose that $m\gamma' = m_1\gamma_1 + \cdots + m_r\gamma_r$ with $m \in \mathbb{N}$, $m_1, \ldots, m_r \in \mathbb{Z}$. Since $\widehat{h}_{A_0}(\gamma') \preceq [K(p) : K]$, $\widehat{h}_{A_0}$ extends to a norm on $\Gamma \otimes \mathbb{R}$ and all norms on the finite-dimensional $\mathbb{R}$-vector space $\Gamma \otimes \mathbb{R}$ are equivalent, we also have that $\max_{i=1,\ldots,r} \frac{|m_i|}{m} \preceq [K(p) : K]$.

For given $N \in \mathbb{N}$, we can find $n \leq N$ and $a_1, \ldots, a_r \in \mathbb{Z}$ such that

$$\max_{i=1,\ldots,r} \left| a_i - \frac{nm_i}{m} \right| \leq \lfloor N^{\frac{1}{r}} \rfloor^{-1}.$$

It follows that

$$\widehat{h}_A \left(\alpha \left(nq - \sum_{i=1}^{r} a_i \gamma_i, 0, \ldots, 0 \right) \right) =$$
$$\widehat{h}_A \left(\alpha \left(n\gamma' - \sum_{i=1}^{r} a_i \gamma_i, 0, \ldots, 0 \right) \right) \leq c_9 \|\alpha\|^2 N^{-\frac{2}{r}},$$

where the constant c_9 depends only on A_0, L_0, A, the choice of symmetric ample line bundle on A as well as of the norm $\|\cdot\|$ on $\mathrm{Hom}(A_0^{r+1}, A) \otimes \mathbb{R}$, and $\gamma_1, \ldots, \gamma_r$. Since $\alpha\left(nq - \sum_{i=1}^{r} a_i \gamma_i, 0, \ldots, 0\right)$ is defined over $K(q)$, it follows from a theorem of Masser ([95], p. 154) that it must be a torsion point the order s of which is polynomially bounded in $[K(p) : K]$ as soon as N exceeds some bound that is polynomial in $[K(p) : K]$ (recall that $\|\alpha\| \preceq [K(p) : K]$).

So we may choose $N \preceq [K(p) : K]$ and $a_1, \ldots, a_r \in \mathbb{Z}$ such that $(sNq - \sum_{i=1}^{r} s a_i \gamma_i, 0, \ldots, 0) \in \ker \alpha$ and therefore $Nq - \sum_{i=1}^{r} a_i \gamma_i \in t + B_1$ for a torsion point t. Furthermore, the a_i satisfy by construction $|a_i| \leq \frac{N|m_i|}{m} + 1 \preceq [K(p) : K]$.

Applying Proposition 9.1 from [70] again, we see that t can be chosen such that its order is polynomially bounded in $[K(p) : K]$. This proves (iii) and thereby the proposition. $\square$

3.5. o-Minimality

We give a brief introduction to the theory of o-minimal structures and define all terms which are relevant in our application. We refer to the book of van den Dries ([180]) for a more thorough treatment of o-minimal structures.

DEFINITION 3.5.1. *A set $A \subset \mathbb{R}^n$ is called semialgebraic if it is a finite union of sets of the form*

$$\{x \in \mathbb{R}^n; f(x) = 0,\ g_1(x) > 0, \ldots, g_s(x) > 0\},$$

where f, $g_1, \ldots, g_s \in \mathbb{R}[X_1, \ldots, X_n]$. A map $f : A \to \mathbb{R}^m$ ($A \subset \mathbb{R}^n$) is called semialgebraic if its graph is semialgebraic.

DEFINITION 3.5.2. *An o-minimal structure $\mathfrak{S}$ (over $(\mathbb{R}, +, -, \cdot, <, 0, 1)$) is a sequence $\mathfrak{S} = (\mathfrak{S}_n)_{n\in\mathbb{N}}$ such that $\mathfrak{S}_n$ is a subset of the power set of $\mathbb{R}^n$ for all $n \in \mathbb{N}$ and the following conditions are satisfied:*

(i) $A, B \in \mathfrak{S}_n \implies A \cup B,\ \mathbb{R}^n \backslash A \in \mathfrak{S}_n$.
(ii) $A \in \mathfrak{S}_n \implies \mathbb{R} \times A \in \mathfrak{S}_{n+1}$.
(iii) $A \in \mathfrak{S}_{n+1} \implies p_n(A) \in \mathfrak{S}_n$, where $p_n : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is the projection onto the first n factors.
(iv) All semialgebraic subsets of $\mathbb{R}^n$ belong to $\mathfrak{S}_n$.
(v) The set $\mathfrak{S}_1$ consists precisely of all finite unions of point sets $\{a\}$ ($a \in \mathbb{R}$) and open intervals (a, b) ($a \in \mathbb{R} \cup \{-\infty\}$, $b \in \mathbb{R} \cup \{\infty\}$).

We call the elements of $\bigcup_{n\in\mathbb{N}} \mathfrak{S}_n$ the definable sets with respect to $\mathfrak{S}$ or simply the definable sets (if $\mathfrak{S}$ is fixed).

Since our uniformization map goes to a product of projective spaces, we need to introduce the notion of a definable space. This notion is treated in more detail by van den Dries in Chapter 10 of [180]. In the following definitions, definability will always mean definability with respect to some fixed o-minimal structure $\mathfrak{S}$.

DEFINITION 3.5.3. *Suppose that $A \subset \mathbb{R}^m$ and $B \subset \mathbb{R}^n$. A map $f : A \to B$ is called definable if its graph*

$$\{(x, f(x));\ x \in A\}$$

is definable.

DEFINITION 3.5.4. *A definable space is a set $D = \cup_{i\in I} U_i$ with I finite together with bijective maps $f_i : U_i \to U_i'$, where $U_i' \subset \mathbb{R}^{m_i}$ is a definable set, such that for all i, j the set $f_i(U_i \cap U_j)$ is definable and open in U_i' and the map $f_j \circ (f_i^{-1})|_{f_i(U_i\cap U_j)} : f_i(U_i \cap U_j) \to f_j(U_i \cap U_j)$ is definable and continuous. A set $X \subset D$ is called definable if $f_i(X \cap U_i)$ is definable for every $i \in I$. We call the f_i charts of D.*

DEFINITION 3.5.5. *Suppose that D_1 and D_2 are definable spaces. A map $F : D_1 \to D_2$ is called a morphism if for every chart $f : U \to U'$ of D_1 and every chart $g : V \to V'$ of D_2 the set $f(U \cap F^{-1}(V))$ is definable and open in U' and the map $g \circ F \circ (f^{-1})|_{f(U\cap F^{-1}(V))}$ is continuous and definable.*

It is easily seen that image and pre-image of a definable set under a definable map or a morphism are definable and that the composition of two definable maps or morphisms is again a definable map or a morphism respectively. A definable map is a morphism with respect to the standard global charts of its domain and its range precisely if it is continuous.

By the Seidenberg-Tarski theorem, the semialgebraic sets themselves form an o-minimal structure (the definable maps of which are the semialgebraic maps). For our purposes, this will not be sufficient and we will have to work in the structure $\mathbb{R}_{\mathrm{an,exp}}$, which contains (among other things) the graph of the exponential function on the real numbers and the graph of the restriction of any analytic function, defined on an open neighbourhood of $[0,1]^n$, to $[0,1]^n$ ($n \in \mathbb{N}$). That this structure is o-minimal and admits analytic cell decomposition is due to van den Dries and Miller (see [181]).

In order to prove our main theorem, we will need that rational points on definable sets are sparse unless there is a "reason" for them not to be sparse in the form of a semialgebraic set, contained in the definable set. This is the famous Pila-Wilkie Theorem. We will use a variant by Habegger and Pila, counting "semirational" points, which is what we will need in the proof.

THEOREM 3.5.6. *(Habegger-Pila) Let $Z \subset \mathbb{R}^m \times \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ be a definable set and $\epsilon > 0$. Let π_1, π_2, and π_3 be the projections onto $\mathbb{R}^m$, $\mathbb{R}^{n_1}$, and $\mathbb{R}^{n_2}$ respectively. There is a constant $c = c(Z, \epsilon) > 0$ with the following property: If $T \geq 1$ and*

$$|\pi_3(\{(y, z_1, z_2) \in Z;\, y = (y_1, \ldots, y_m) \in \mathbb{Q}^m,\ \max_{j=1,\ldots,m} H(y_j) \leq T\})| > cT^\epsilon,$$

there exists a continuous and definable function $\delta : [0,1] \to Z$ such that the following properties hold:

(i) *The composition $\pi_1 \circ \delta : [0,1] \to \mathbb{R}^m$ is semialgebraic and its restriction to $(0,1)$ is real analytic.*
(ii) *The composition $\pi_3 \circ \delta : [0,1] \to \mathbb{R}^{n_2}$ is non-constant.*
(iii) *If the o-minimal structure admits analytic cell decomposition, the restriction of δ to $(0,1)$ is real analytic.*

PROOF. This is a special case of Corollary 7.2 in [70] with $k = \mathbb{Q}$, $\mathbb{R}^n = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, and

$$\Sigma = \{(y, z_1, z_2) \in Z;\, y = (y_1, \ldots, y_m) \in \mathbb{Q}^m,\ \max_{j=1,\ldots,m} H(y_j) \leq T\}.$$

A priori, the corollary only provides δ such that $(\pi_2, \pi_3) \circ \delta$ is non-constant. Going through its proof, we see however that δ can actually be chosen such that $\pi_3 \circ \delta$ is non-constant. Note that we do not need the additional uniformity in families that the corollary provides. $\square$

3.6. Definability

In order to be able to use the powerful o-minimality result from the last section, we must show that our analytic uniformization of $\mathfrak{A}_{g,(2l,l)}(\mathbb{C})$ is definable when restricted to a suitable set. In order to be able to speak of e.g. definable or semialgebraic subsets of $\mathbb{C}$ or $\mathbb{H}_g$, we will always identify $\mathbb{C}$ with $\mathbb{R}^2$ and $\mathrm{M}_g(\mathbb{C})$ with $\mathbb{R}^{2g^2}$ by identifying $u + v\sqrt{-1} \in \mathbb{C}$ with $(u, v) \in \mathbb{R}^2$. The following important proposition is due to Peterzil-Starchenko. Note that $\mathbb{P}^{l^g-1}(\mathbb{C})$ is a definable space with respect to its standard atlas.

PROPOSITION 3.6.1. *(Peterzil-Starchenko) The map* $\exp : \mathbb{H}_g \times \mathbb{C}^g \to \mathfrak{A}_{g,(2l,l)}(\mathbb{C}) \subset \mathbb{P}^{l^g-1}(\mathbb{C}) \times \mathbb{P}^{l^g-1}(\mathbb{C})$*, defined as in Proposition 3.2.1, has the following properties:*

(i) There is an open subset U of $\mathbb{H}_g \times \mathbb{C}^g$ such that the restriction of $\exp$ *to U is a morphism of definable spaces in $\mathbb{R}_{\mathrm{an,exp}}$ and U contains the set*

$$\{(\tau, \Omega_\tau x) \in F \times \mathbb{C}^g; x \in [0,1)^{2g}\},$$

where F is a Siegel fundamental domain for the congruence subgroup $G(l, 2l)$ of $\mathrm{Sp}_{2g}(\mathbb{Z})$.

(ii) The map $\exp|_U$ *is surjective.*

PROOF. Going back to the proof of Proposition 3.2.1, we see that it suffices to show that the function ϕ as defined there is definable, when restricted to an open set that contains

$$\{(l\tau, \Omega_\tau lx) \in \mathbb{H}_g \times \mathbb{C}^g; \tau \in F, x \in [0,1)^{2g}\}.$$

This is a consequence of Corollary 7.10(1) in [128] with $D = lI_g$ since F consists of finitely many translates of the Siegel fundamental domain and $l\tau = M[\tau]$, where

$$M = \begin{pmatrix} \sqrt{l} I_g & 0 \\ 0 & \frac{1}{\sqrt{l}} I_g \end{pmatrix} \in \mathrm{Sp}_{2g}(\mathbb{R}).$$

As exp is clearly continous, we deduce (i).

Next, we deduce (ii) from Proposition 3.2.1(ii) since U contains at least one element of each orbit of the action of $G(l, 2l) \ltimes \mathbb{Z}^{2g}$ on $\mathbb{H}_g \times \mathbb{C}^g$. □

3.7. Functional transcendence

Let $S \subset A_{g,(2l,l)}$ be an irreducible smooth locally closed curve, set $\mathcal{A} = \pi^{-1}(S)$, and let $\mathcal{C} \subset \mathcal{A}$ be an irreducible closed curve. Let exp be as in Proposition 3.2.1. Once we have used the Habegger-Pila theorem to find a semialgebraic obstruction, the following theorem (known as "Ax of log type"), which is due to Gao, will allow us to conclude that $\mathcal{C}$ is contained in an irreducible variety as described in Theorem 3.1.2 of suitable codimension.

Recall that ξ is the generic point of S and $\left(\mathcal{A}_\xi^{\overline{\bar{\mathbb{Q}}(S)}/\bar{\mathbb{Q}}}, \mathrm{Tr}\right)$ is the $\overline{\bar{\mathbb{Q}}(S)}/\bar{\mathbb{Q}}$-trace of $\mathcal{A}_\xi$. In this section, we will use subscripts to denote the base change of varieties and morphisms.

THEOREM 3.7.1. *(Gao) Suppose that $\pi(\mathcal{C}) = S$. Let $\tilde{Y}$ be an arbitrary complex analytic irreducible component of* $\exp^{-1}(\mathcal{C}(\mathbb{C}))$*. Then exactly one complex analytic irreducible component of the intersection of the Zariski closure of $\tilde{Y}$ in $\mathrm{M}_g(\mathbb{C}) \times \mathbb{C}^g$ with $\mathbb{H}_g \times \mathbb{C}^g$ contains $\tilde{Y}$. Furthermore, the intersection of this component with* $\exp^{-1}(\mathcal{A}(\mathbb{C}))$ *maps under* exp *onto the complex points of a subvariety $\mathcal{W}$ of $\mathcal{A}$ containing $\mathcal{C}$ such that over $\overline{\bar{\mathbb{Q}}(S)}$, every irreducible component of $\mathcal{W}_\xi$ is a translate of an abelian subvariety of $\mathcal{A}_\xi$ by a point in* $(\mathcal{A}_\xi)_{\mathrm{tors}} + \mathrm{Tr}\left(\mathcal{A}_\xi^{\overline{\bar{\mathbb{Q}}(S)}/\bar{\mathbb{Q}}}(\bar{\mathbb{Q}})\right)$.

One irreducible component of the variety $\mathcal{W}$ in Theorem 3.7.1 is the variety the existence of which Theorem 3.1.2 postulates. Our statement of the theorem differs from Gao's in the terminology that we use. Before we can prove that our version follows from Gao's version, we need to introduce Gao's terminology: We follow the exposition in [53]. Recall that an abelian subscheme $\mathcal{B}$ of $\mathcal{A}_{\mathbb{C}}$ is an irreducible subgroup scheme of $\mathcal{A}_{\mathbb{C}} \to S_{\mathbb{C}}$ which is proper and flat over $S_{\mathbb{C}}$ and dominates $S_{\mathbb{C}}$. An irreducible subvariety $\mathcal{Z}$ of $\mathcal{A}_{\mathbb{C}}$ is called a generically special subvariety of sg type if there exists a finite cover $S' \to S_{\mathbb{C}}$, inducing a morphism $\rho : \mathcal{A}' = \mathcal{A}_{\mathbb{C}} \times_{S_{\mathbb{C}}} S' \to \mathcal{A}_{\mathbb{C}}$ such that $\mathcal{Z} = \rho(\sigma' + \sigma_0' + \mathcal{B}')$, where $\mathcal{B}'$ is an abelian subscheme of $\mathcal{A}'$, σ' is a torsion section of $\mathcal{A}'$, and σ_0' is a constant section of $\mathcal{A}'$, i.e. the composition of a section $S' \to C' \times S', s \mapsto (q, s)$ (C' an abelian variety over $\mathbb{C}$, $q \in C'(\mathbb{C})$) with an isomorphism between $C' \times S'$ and an abelian subscheme of $\mathcal{A}'$. We can now prove Theorem 3.7.1.

Proof. We apply Theorem 8.1 from [52] to the connected mixed Shimura variety $\mathfrak{A}_{g,(2l,l)}(\mathbb{C})$ with uniformization map $\exp : \mathbb{H}_g \times \mathbb{C}^g \to \mathfrak{A}_{g,(2l,l)}(\mathbb{C})$ and subvariety Y equal to the Zariski closure of $\mathcal{C}(\mathbb{C})$ in $\mathfrak{A}_{g,(2l,l)}(\mathbb{C})$. As $Y(\mathbb{C})\backslash\mathcal{C}(\mathbb{C})$ is finite, the closure with respect to the Euclidean topology of $\tilde{Y}$ is a complex analytic irreducible component of $\exp^{-1}(Y(\mathbb{C}))$. Together with Theorem 8.1 in [52], this implies that exactly one complex analytic irreducible component of the intersection of the Zariski closure of $\tilde{Y}$ in $\mathrm{M}_g(\mathbb{C}) \times \mathbb{C}^g$ with $\mathbb{H}_g \times \mathbb{C}^g$ contains $\tilde{Y}$ and that this component maps onto the complex points of a weakly special subvariety $\tilde{\mathcal{W}}$ of $\left(\mathfrak{A}_{g,(2l,l)}\right)_{\mathbb{C}}$ and that $\tilde{\mathcal{W}}$ is the smallest weakly special subvariety containing Y. As a weakly special subvariety, $\tilde{\mathcal{W}}$ is irreducible. By Proposition 5.3 in [53] (cf. Proposition 1.1 and 3.3 in [51]) the variety $\tilde{\mathcal{W}}$ is a generically special subvariety of sg type of the abelian scheme $\pi^{-1}(\pi(\tilde{\mathcal{W}})) \to \pi(\tilde{\mathcal{W}})$, where this term is defined analogously for $\pi^{-1}(\pi(\tilde{\mathcal{W}})) \to \pi(\tilde{\mathcal{W}})$ as for $\mathcal{A}_{\mathbb{C}} \to S_{\mathbb{C}}$ (see Definition 1.5 in [53]).

A priori, $\tilde{\mathcal{W}}$ is defined over $\mathbb{C}$, but since it is the smallest such weakly special subvariety, images of weakly special subvarieties under $\mathrm{Aut}(\mathbb{C}/\bar{\mathbb{Q}})$ as well as irreducible components of intersections of weakly special subvarieties are weakly special, and $\mathcal{C}$ and hence Y are defined over $\bar{\mathbb{Q}}$, it must be defined over $\bar{\mathbb{Q}}$. We consider $\tilde{\mathcal{W}}$ as a variety over $\bar{\mathbb{Q}}$ and set $\mathcal{W} = \tilde{\mathcal{W}} \cap \mathcal{A}$. This is a subvariety of $\mathcal{A}$ that contains $\mathcal{C}$.

Let L be an algebraic closure of the function field of $S_{\mathbb{C}}$. We identify $\overline{\bar{\mathbb{Q}}(S)}$ with the algebraic closure of $\bar{\mathbb{Q}}(S)$ in L. The $L/\mathbb{C}$-trace of $(\mathcal{A}_\xi)_L$ coincides with the base change of the $\mathbb{C}\overline{\bar{\mathbb{Q}}(S)}/\mathbb{C}$-trace of $(\mathcal{A}_\xi)_{\mathbb{C}\overline{\bar{\mathbb{Q}}(S)}}$, which coincides with the base change of the $\overline{\bar{\mathbb{Q}}(S)}/\bar{\mathbb{Q}}$-trace of $\mathcal{A}_\xi$ by Theorem 6.8 in [33]. As $\tilde{\mathcal{W}}$ is generically special of sg type (as a variety over $\mathbb{C}$), it follows from the universal property of the trace that every irreducible component of $(\mathcal{W}_\xi)_L$ is a translate of an abelian subvariety of $(\mathcal{A}_\xi)_L$ by a point in $(\mathcal{A}_\xi)_{\mathrm{tors}} + \mathrm{Tr}_L\left(\mathcal{A}_\xi^{\overline{\bar{\mathbb{Q}}(S)}/\bar{\mathbb{Q}}}(\mathbb{C})\right)$.

For every torsion point t of $\mathcal{A}_\xi$, the subvariety $\mathrm{Tr}_L^{-1}\left((t + \mathcal{W}_\xi)_L\right)$ of $\left(\mathcal{A}_\xi^{\overline{\bar{\mathbb{Q}}(S)}/\bar{\mathbb{Q}}}\right)_L$ is then defined both over $\mathbb{C}$ and $\overline{\bar{\mathbb{Q}}(S)}$ since all abelian subvarieties and torsion points of $\left(\mathcal{A}_\xi^{\overline{\bar{\mathbb{Q}}(S)}/\bar{\mathbb{Q}}}\right)_L$ are defined over $\bar{\mathbb{Q}}$. Hence, $\mathrm{Tr}_L^{-1}\left((t + \mathcal{W}_\xi)_L\right)$ is defined over the intersection of these two fields in L, which is equal to $\bar{\mathbb{Q}}$. Therefore, the irreducible components of $\mathcal{W}_\xi$ are in fact translates of abelian subvarieties by points in $(\mathcal{A}_\xi)_{\mathrm{tors}}+\mathrm{Tr}\left(\mathcal{A}_\xi^{\overline{\bar{\mathbb{Q}}(S)}/\bar{\mathbb{Q}}}(\bar{\mathbb{Q}})\right)$. The theorem follows.

Note that Proposition 5.3 in [53] only applies to the universal family of principally polarized abelian varieties with symplectic level l-structure, but the same statement can be proven analogously for any connected mixed Shimura variety of Kuga type coming from a neat congruence subgroup, so in particular for $\mathfrak{A}_{g,(2l,l)}(\mathbb{C})$ (see Proposition 1.2.14 and Corollary 1.2.15 in Gao's dissertation [50]). One could also apply Proposition 5.3 from [53] to an irreducible component of the preimage of $\tilde{\mathcal{W}}(\mathbb{C})$ under the canonical Shimura morphism from the universal family of principally polarized abelian varieties with symplectic level $2l$-structure to $\mathfrak{A}_{g,(2l,l)}(\mathbb{C})$. □

3.8. Proof of Theorem 3.1.2, Theorem 3.1.3, and Corollary 3.1.4

3.8.1. Proof of Theorem 3.1.2. We assume that $\mathcal{A}_\Gamma^{[k]}\cap\mathcal{C}$ is infinite and want to show that $\mathcal{C}$ is contained in an irreducible subvariety $\mathcal{W}$ of the form described in Theorem 3.1.2.

3.8.1.1. *Reduction to the universal family.*

LEMMA 3.8.1. *We can assume without loss of generality that $S\subset A_{g,(2l,l)}$ is a smooth irreducible locally closed curve (not necessarily closed in $A_{g,(2l,l)}$) and $\mathcal{A}=\pi^{-1}(S)$.*

PROOF. For l big enough, the scheme $A_{g,(2l,l)}$ with the family of abelian varieties $\mathfrak{A}_{g,(2l,l)}\to A_{g,(2l,l)}$ is the fine moduli scheme of principally polarized abelian varieties of dimension g with level structure "between l and $2l$". For the precise moduli interpretation, see [120], Appendix to Chapter 7, Section B. In particular, if our family is a pull-back of the universal family of principally polarized abelian varieties of dimension g with symplectic level $2l$-structure, it will automatically also be a pull-back of $\mathfrak{A}_{g,(2l,l)}\to A_{g,(2l,l)}$.

Let therefore $\mathcal{A}\to S$ for the moment be an arbitrary abelian scheme over an irreducible smooth curve of relative dimension g. If ξ is the generic point of S, then the abelian variety $\mathcal{A}_\xi$ is isogenous to a principally polarized abelian variety $\tilde{A}$. The abelian variety $\tilde{A}$ as well as the isogeny are defined over some finite extension F of $\bar{\mathbb{Q}}(S)$. After replacing S by a finite cover $S'\to S$ and $\mathcal{A}$ by its pull-back under that cover, we may assume that $F=\bar{\mathbb{Q}}(S)$. We can replace S by a finite cover since an irreducible subvariety $\mathcal{W}\subset\mathcal{A}\times_S\mathcal{S}'$ as described in Theorem 3.1.2 projects to an irreducible subvariety of $\mathcal{A}$ of the same form. By Theorem 3 in Section 1.4 of [25], there exists a Néron model $\tilde{\mathcal{A}}$ of $\tilde{A}$ over S as defined in Definition 1 in Section 1.2 of [25]. By the universal property of the Néron model, we obtain an S-morphism $\mathcal{A}\to\tilde{\mathcal{A}}$ which extends the isogeny between $\mathcal{A}_\xi$ and $\tilde{A}$.

By Theorem 3 in Section 1.4 of [25], there is a Zariski open subset $\tilde{S}$ of S such that $\tilde{\mathcal{A}}\times_S\tilde{S}$ is an abelian scheme over $\tilde{S}$. Since $\tilde{S}$ is smooth, it follows as in [45], p. 6, that the abelian scheme $\tilde{\mathcal{A}}\times_S\tilde{S}$ is principally polarized, i.e. admits an isomorphism of group schemes over $\tilde{S}$ to its dual abelian scheme such that on each geometric fiber the corresponding base change of the isomorphism is induced by an ample line bundle on that fiber. The morphism between $\mathcal{A}$ and $\tilde{\mathcal{A}}$ that extends the isogeny between $\mathcal{A}_\xi$ and $\tilde{A}$ is dominant and proper, hence surjective, so its restriction to each fiber over a point in $\tilde{S}$ is an isogeny. We see that it suffices to prove the theorem for $\tilde{\mathcal{A}} \times_S \tilde{S} \to \tilde{S}$, hence we can assume that $\mathcal{A}$ is a principally polarized abelian scheme.

We can then add symplectic level $2l$-structure to the family $\mathcal{A} \to S$ by taking a finite cover of S (corresponding to the finite field extension of $\bar{\mathbb{Q}}(S)$ that is obtained by adding the $2l$-torsion points of the generic fiber).

Having done this, there is a cartesian diagram

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{i} & \mathfrak{A}_{g,(2l,l)} \\ \downarrow & & \downarrow \\ S & \xrightarrow{i_S} & A_{g,(2l,l)} \end{array},$$

where the morphisms i and i_S are defined over $\bar{\mathbb{Q}}$. This is a consequence of Theorem 7.9 in [120] that asserts the existence of a fine moduli space for principally polarized abelian varieties of dimension g with full level l-structure for l big enough (in fact, $l \geq 3$ suffices). The family $\mathcal{A}$ is then a pull-back of the universal family with symplectic level $2l$-structure and therefore also of the family $\mathfrak{A}_{g,(2l,l)} \to A_{g,(2l,l)}$ (cf. [120], Appendix to Chapter 7, Section B). For every $s \in S$, the restriction $i|_{\mathcal{A}_s}$ is an isomorphism between $\mathcal{A}_s$ and $i(\mathcal{A}_s)$.

If the family $\mathcal{A}$ is not isotrivial, as we suppose in our theorem, the map i_S is non-constant, so has finite fibers, and therefore i has finite fibers as well. Thus, the curve $i(\mathcal{C})$ must intersect the enlarged isogeny orbit in infinitely many points as well. If $\mathcal{W} \subset i(\mathcal{A})$ is of the form described in Theorem 3.1.2, then so is every irreducible component of $i^{-1}(\mathcal{W}) \subset \mathcal{A}$ that dominates S; therefore it suffices to prove our theorem for $i(\mathcal{A}) \to i_S(S)$. We can even pass to a Zariski open smooth subset of $i_S(S)$ (we use that $i(\mathcal{C})$ intersects every fiber in only finitely many points). This proves the lemma. $\square$

3.8.1.2. *Producing many points of bounded height.* We now return to subfamilies of $\mathfrak{A}_{g,(2l,l)} \to A_{g,(2l,l)}$ of the form $\pi^{-1}(S) \to S$ with S smooth, irreducible, and locally closed. We will keep the same notation until the end of the proof.

We have

$$\sup_{p \in \mathcal{A}_\Gamma^{[k]} \cap \mathcal{C}} [K(p) : K] = \infty,$$

since otherwise $\{\pi(p); p \in \mathcal{A}_\Gamma^{[k]} \cap \mathcal{C}\}$ would be a subset of S of bounded degree and hence bounded height by Lemma 3.4.1. By Northcott's theorem, this set would be finite and hence, $\mathcal{A}_\Gamma^{[k]} \cap \mathcal{C}$ would be finite as well since $\mathcal{C}$ intersects every fiber of π in only finitely many points.

For each $s \in S$ such that A_0 and $\mathcal{A}_s$ are isogenous, let $\phi_s : A_0 \to \mathcal{A}_s$ be the isogeny furnished by Corollary 3.3.4. We choose a point $p \in \mathcal{A}_\Gamma^{[k]} \cap \mathcal{C}$. Thanks to Lemma 3.2.2, we can write $p = \phi_{\pi(p)}(\gamma + b)$ for some $\gamma \in \Gamma$, $b \in B_0$ with B_0 an abelian subvariety of A_0 of codimension $\geq k$. We set $s = \pi(p)$, $d = [K(p) : K]$. By the above, we can make d arbitrarily big with the right choice of p. If σ is an element of $\mathrm{Gal}(\bar{\mathbb{Q}}/K)$, then it follows that $\sigma(p) = \sigma(\phi_s)(\sigma(\gamma) + \sigma(b))$, where σ acts on algebraic points and maps in the usual way.

As $\mathcal{C}$ and S are defined over K, the points $\sigma(p)$ and $\sigma(s)$ lie again on $\mathcal{C}$ and S respectively. Note that the addition morphism $A_0 \times A_0 \to A_0$ and the inversion morphism $A_0 \to A_0$ are both defined over K; in particular, σ fixes the zero element of A_0. Furthermore, it sends the zero element of $\mathcal{A}_s$ to the zero element of $\mathcal{A}_{\sigma(s)}$. It follows that the map $\sigma(\phi_s)$ is an isogeny between $\sigma(A_0) = A_0$ and $\sigma(\mathcal{A}_s) = \mathcal{A}_{\sigma(s)}$ with kernel $\sigma(\ker \phi_s)$ and therefore has degree $\deg \sigma(\phi_s) = \deg \phi_s$. Since we have assumed that all endomorphisms of A_0 are defined over K, we have $\sigma(B_0) = B_0$.

Finally, if $N \in \mathbb{N}$ is minimal such that $N\gamma = a_1\gamma_1 + \cdots + a_r\gamma_r$ with rational integers $a_1, \ldots, a_r$, then

$$N\sigma(\gamma) = \sigma(N\gamma) = a_1\sigma(\gamma_1) + \cdots + a_r\sigma(\gamma_r) = a_1\gamma_1 + \cdots + a_r\gamma_r.$$

It follows that $\sigma(\gamma) \in \Gamma$ and hence $\sigma(p) \in \mathcal{A}_\Gamma^{[k]} \cap \mathcal{C}$. By Lemma 3.2.2, there exist $\gamma_\sigma \in \Gamma$, an abelian subvariety B_σ of A_0 of codimension $\geq k$, and $b_\sigma \in B_\sigma$ such that $\phi_{\sigma(s)}(\gamma_\sigma + b_\sigma) = \sigma(p)$, where $\phi_{\sigma(s)}$ is the isogeny chosen in Corollary 3.3.4 and $\deg \phi_{\sigma(s)} \leq \deg \sigma(\phi_s) = \deg \phi_s$. Indeed, we must have $\deg \phi_{\sigma(s)} = \deg \phi_s$ since otherwise $\sigma^{-1}\left(\phi_{\sigma(s)}\right)$ would be an isogeny between A_0 and $\mathcal{A}_s$ of degree less than $\deg \phi_s$, a contradiction. We can choose γ_σ and b_σ as in Proposition 3.4.3.

Thus, we get d different points $\sigma(p)$ in $\mathcal{A}_\Gamma^{[k]} \cap \mathcal{C}$. Each of these points has some pre-image (τ_σ, p_σ) in U under $\exp|_U$ because of Proposition 3.6.1(ii), where exp and U are defined as in that same proposition. From the proof of Proposition 3.6.1(ii), we see that we can choose τ_σ in a Siegel fundamental domain for $G(l, 2l)$ and p_σ in a corresponding fundamental parallelogram for the lattice $\tau_\sigma\mathbb{Z}^g + \mathbb{Z}^g$, i.e. $p_\sigma = \Omega_{\tau_\sigma} x_\sigma$ with $x_\sigma \in [0, 1)^{2g}$.

The isogeny $\phi_{\sigma(s)}$ pulls back under $\exp(\tau_\sigma, \cdot)$ and $\exp_0$ to a linear map from $\mathbb{C}^g$ to itself, given by some matrix $\alpha_\sigma \in \mathrm{GL}_g(\mathbb{C})$ such that $\alpha_\sigma(\Omega_{\tau_0}\mathbb{Z}^{2g}) \subset \Omega_{\tau_\sigma}\mathbb{Z}^{2g}$ is a subgroup of index $\deg \phi_{\sigma(s)} = \deg \phi_s$.

Therefore, there is a matrix $\beta_\sigma \in \mathrm{M}_{2g}(\mathbb{Z}) \cap \mathrm{GL}_{2g}(\mathbb{Q})$ (the rational representation of $\phi_{\sigma(s)}$ with respect to the given uniformizations) satisfying

$$\begin{pmatrix} \alpha_\sigma & 0 \\ 0 & \overline{\alpha_\sigma} \end{pmatrix} \begin{pmatrix} \Omega_{\tau_0} \\ \Omega_{\overline{\tau_0}} \end{pmatrix} = \begin{pmatrix} \Omega_{\tau_\sigma} \\ \Omega_{\overline{\tau_\sigma}} \end{pmatrix} \beta_\sigma.$$

We have $\deg \phi_{\sigma(s)} = |\Delta_\sigma|$, where $\Delta_\sigma := \det \beta_\sigma$.

In fact, the determinant is positive, as it follows from the above that

$$|\det \alpha_\sigma|^2 (2\sqrt{-1})^g \det(\operatorname{Im} \tau_0) = (2\sqrt{-1})^g \det(\operatorname{Im} \tau_\sigma)(\det \beta_\sigma).$$

Therefore, we get $\Delta_\sigma = \deg \phi_{\sigma(s)} = \deg \phi_s$ and $\Delta := \Delta_\sigma$ is independent of σ.

We can write

$$\beta_\sigma = \begin{pmatrix} \beta_{\sigma,1} & \beta_{\sigma,2} \\ \beta_{\sigma,3} & \beta_{\sigma,4} \end{pmatrix}$$

with $\beta_{\sigma,j} \in \mathrm{M}_g(\mathbb{Z})$ $(j = 1, \ldots, 4)$. It then follows from the above that

$$\alpha_\sigma^{-1}\Omega_{\tau_\sigma} = \Omega_{\tau_0}(\beta_\sigma)^{-1} \tag{3.8.1}$$

and that

$$\alpha_\sigma\tau_0 = \tau_\sigma\beta_{\sigma,1} + \beta_{\sigma,3}, \quad \alpha_\sigma = \tau_\sigma\beta_{\sigma,2} + \beta_{\sigma,4},$$

whence we obtain

$$\tau_0 = (\tau_\sigma\beta_{\sigma,1} + \beta_{\sigma,3})^t(\tau_\sigma\beta_{\sigma,2} + \beta_{\sigma,4})^{-t} = (\beta_{\sigma,1}^t\tau_\sigma + \beta_{\sigma,3}^t)(\beta_{\sigma,2}^t\tau_\sigma + \beta_{\sigma,4}^t)^{-1}. \tag{3.8.2}$$

The point $p_\sigma \in \mathbb{C}^g$ satisfies

$$\exp_0(\alpha_\sigma^{-1} p_\sigma) \in \phi_{\sigma(s)}^{-1}(\sigma(p)) = \gamma_\sigma + b_\sigma + \ker \phi_{\sigma(s)}$$

and it follows thanks to $|\ker \phi_{\sigma(s)}| = \deg \phi_{\sigma(s)} = \Delta$ that

$$\exp_0(N_\sigma \Delta \alpha_\sigma^{-1} p_\sigma) = N_\sigma \Delta \gamma_\sigma + N_\sigma \Delta b_\sigma = \Delta \left(a_{\sigma,1}\gamma_1 + \cdots + a_{\sigma,r}\gamma_r \right) + N_\sigma \Delta b_\sigma,$$

where $N_\sigma \in \mathbb{N}$ is minimal such that $N_\sigma \gamma_\sigma \in \mathbb{Z}\gamma_1 + \cdots + \mathbb{Z}\gamma_r$ and $a_{\sigma,1}, \ldots, a_{\sigma,r} \in \mathbb{Z}$.

As the kernel of $\exp_0$ is $\Omega_{\tau_0}\mathbb{Z}^{2g}$, we deduce that

$$\Delta(N_\sigma \alpha_\sigma^{-1} p_\sigma - \tilde{\gamma}_\sigma - N_\sigma \tilde{b}_\sigma) = \Omega_{\tau_0} R_\sigma, \tag{3.8.3}$$

where $R_\sigma \in \mathbb{Z}^{2g}$, $\tilde{b}_\sigma = \Omega_{\tau_0}\tilde{y}_\sigma$ satisfies $\exp_0(\tilde{b}_\sigma) = b_\sigma$ ($\tilde{y}_\sigma \in [0,1)^{2g}$), and $\tilde{\gamma}_\sigma = a_{\sigma,1}\tilde{\gamma}_1 + \cdots + a_{\sigma,r}\tilde{\gamma}_r$ with $\tilde{\gamma}_i = \Omega_{\tau_0} u_i$, $u_i \in [0,1)^{2g}$, and $\exp_0(\tilde{\gamma}_i) = \gamma_i$ ($i = 1, \ldots, r$).

It now follows from Proposition 3.4.3(ii) and Lemma 3.3.5 that there exist a matrix $H_\sigma \in \mathrm{M}_{2g\times 2(g-k)}(\mathbb{Z})$ and $y_\sigma \in [0,1)^{2(g-k)}$ such that $\tilde{y}_\sigma - H_\sigma y_\sigma \in \mathbb{Z}^{2g}$, $\Omega_{\tau_0} H_\sigma$ has rank at most $g-k$, and

$$\|H_\sigma\| \preceq [K(p) : K] = d. \tag{3.8.4}$$

After replacing R_σ by $R_\sigma + \Delta N_\sigma(\tilde{y}_\sigma - H_\sigma y_\sigma)$, we can assume that $\tilde{y}_\sigma = H_\sigma y_\sigma$. Of course, we then no longer necessarily have $\tilde{y}_\sigma \in [0,1)^{2g}$.

LEMMA 3.8.2. *With notation as above, we have*

$$\max\{|a_{\sigma,1}|, \ldots, |a_{\sigma,r}|, N_\sigma, \|R_\sigma\|, \|\beta_\sigma\|, \|H_\sigma\|\} \preceq d$$

for every $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/K)$.

PROOF. The bound for $\|H_\sigma\|$ has just been established. It follows from Corollary 3.3.4(ii) and Proposition 3.4.3(i) that

$$\|\beta_\sigma\| \preceq \deg \phi_{\sigma(s)} = \deg \phi_s \preceq d. \tag{3.8.5}$$

The matrix $\begin{pmatrix} \Omega_{\tau_0} \\ \Omega_{\overline{\tau_0}} \end{pmatrix}$ is invertible and we have

$$\|R_\sigma\| \leq \left\| \begin{pmatrix} \Omega_{\tau_0} \\ \Omega_{\overline{\tau_0}} \end{pmatrix}^{-1} \right\| \left\| \begin{pmatrix} \Omega_{\tau_0} \\ \Omega_{\overline{\tau_0}} \end{pmatrix} R_\sigma \right\|. \tag{3.8.6}$$

It also follows from (3.8.3) that

$$\|\Omega_{\tau_0} R_\sigma\| = \|\Omega_{\overline{\tau_0}} R_\sigma\| \leq \Delta \left(N_\sigma \|\alpha_\sigma^{-1} p_\sigma\| + (N_\sigma \|H_\sigma\| + |a_{\sigma,1}| + \cdots + |a_{\sigma,r}|)\|\Omega_{\tau_0}\| \right), \tag{3.8.7}$$

since $\tilde{b}_\sigma = \Omega_{\tau_0} H_\sigma y_\sigma$ with $y_\sigma \in [0,1)^{2(g-k)}$ and $\tilde{\gamma}_i = \Omega_{\tau_0} u_i$ with $u_i \in [0,1)^{2g}$ ($i = 1, \ldots, r$).

Furthermore, we know that

$$\alpha_\sigma^{-1} p_\sigma = \alpha_\sigma^{-1} \Omega_{\tau_\sigma} x_\sigma$$

with $x_\sigma \in [0,1)^{2g}$. Using (3.8.1), we deduce that

$$\alpha_\sigma^{-1} p_\sigma = \Omega_{\tau_0} (\beta_\sigma)^{-1} x_\sigma.$$

Therefore, we can estimate very crudely

$$\|\alpha_\sigma^{-1} p_\sigma\| \leq \|\Omega_{\tau_0}\| \|(\beta_\sigma)^{-1}\| \|x_\sigma\| \preceq \|\beta_\sigma\|. \tag{3.8.8}$$

We know thanks to Proposition 3.4.3(iii) that

$$\max\{|a_{\sigma,1}|,\ldots,|a_{\sigma,r}|,N_\sigma\} \preceq [K(\sigma(p)):K] = [K(p):K] = d.$$

Combining this with (3.8.4), (3.8.5), (3.8.6), (3.8.7), and (3.8.8), we deduce that

$$\|R_\sigma\| \preceq d, \tag{3.8.9}$$

where the implicit constants are independent of p, σ, and d. $\square$

3.8.1.3. *Application of the point-counting theorem.* From now on, "definable" will always mean "definable in the o-minimal structure $\mathbb{R}_{\mathrm{an,exp}}$". Let exp and U be defined as in Proposition 3.6.1. The set $X = \exp|_U^{-1}(\mathcal{C}(\mathbb{C})) \subset \mathbb{H}_g \times \mathbb{C}^g$ is definable since $\mathcal{C}(\mathbb{C})$ is semialgebraic, being a quasi-projective algebraic curve, and $\exp|_U$ is definable by Proposition 3.6.1(i).

LEMMA 3.8.3. *There exists a non-constant real analytic map $\alpha : (0,1) \to X$ such that the transcendence degree over $\mathbb{C}$ of the field generated by its complex coordinate functions is at most $g-k+1$.*

PROOF. Consider the definable set

$$\begin{aligned} Z = \{&(A_1,\ldots,A_r,M,R,B_1,B_2,B_3,B_4,H,A,y,\tau,x) \in \mathbb{R}^{r+1+2g} \times \mathrm{M}_g(\mathbb{R})^4 \\ &\times \mathrm{M}_{2g\times 2(g-k)}(\mathbb{R}) \times \mathrm{GL}_g(\mathbb{C}) \times \mathbb{R}^{2(g-k)} \times \mathbb{H}_g \times \mathbb{R}^{2g}; (\tau,\Omega_\tau x) \in X, B = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix}, \\ &\det B > 0, \det(B_2^t\tau + B_4^t) \neq 0, \tau_0(B_2^t\tau + B_4^t) = B_1^t\tau + B_3^t, \\ &\Omega_{\tau_0}H \text{ has rank at most } g-k, M > 0, A^{-1}\Omega_\tau = \Omega_{\tau_0}B^{-1} \\ &\Omega_{\tau_0}R = (\det B)\left(M(\Omega_{\tau_0}B^{-1}x - \Omega_{\tau_0}Hy) - (A_1\tilde{\gamma}_1 + \cdots + A_r\tilde{\gamma}_r)\right)\}. \end{aligned}$$

What we have done so far, in particular (3.8.1), (3.8.2), and (3.8.3), shows that Z contains d points

$$(a_{\sigma,1},\ldots,a_{\sigma,r},N_\sigma,R_\sigma,\beta_{\sigma,1},\beta_{\sigma,2},\beta_{\sigma,3},\beta_{\sigma,4},H_\sigma,\alpha_\sigma,y_\sigma,\tau_\sigma,x_\sigma)$$

such that $a_{\sigma,1},\ldots,a_{\sigma,r} \in \mathbb{Z}$ and $N_\sigma \in \mathbb{N}$, $\beta_{\sigma,1}$, $\beta_{\sigma,2}$, $\beta_{\sigma,3}$, $\beta_{\sigma,4} \in \mathrm{M}_g(\mathbb{Z})$, $H_\sigma \in \mathrm{M}_{2g\times 2(g-k)}(\mathbb{Z})$, and $R_\sigma \in \mathbb{Z}^{2g}$. By Lemma 3.8.2, there is a constant κ, independent of p, σ, and d, such that

$$\max\{|a_{\sigma,1}|,\ldots,|a_{\sigma,r}|,N_\sigma,\|R_\sigma\|,\|\beta_\sigma\|,\|H_\sigma\|\} \leq d^\kappa =: T$$

for d large enough.

We choose $\epsilon = (2\kappa)^{-1}$ and call the set of these d points Σ. By making d large enough, we can ensure that $d = |\pi_3(\Sigma)| > c(Z,\epsilon)T^\epsilon$, where $c(Z,\epsilon)$ is the constant from Theorem 3.5.6 and π_3 is the projection $\pi_3 : Z \to \mathbb{H}_g \times \mathbb{R}^{2g}$ (note that the (τ_σ,x_σ) are all different since the $\sigma(p) = \exp(\tau_\sigma,\Omega_{\tau_\sigma}x_\sigma)$ are).

In the framework of Theorem 3.5.6 with projection maps $\pi_1 : Z \to \mathbb{R}^{r+1+2g} \times \mathrm{M}_g(\mathbb{R})^4 \times \mathrm{M}_{2g\times 2(g-k)}(\mathbb{R})$, $\pi_2 : Z \to \mathrm{GL}_g(\mathbb{C}) \times \mathbb{R}^{2(g-k)}$, and π_3, we have that

$$\Sigma \subset \{(y,z_1,z_2) \in Z;\, y = (y_1,\ldots,y_m) \in \mathbb{Q}^m,\ \max_{j=1,\ldots,m} H(y_j) \leq T\}$$

and $|\pi_3(\Sigma)| > cT^\epsilon$. Hence, we can apply Theorem 3.5.6 to Z and obtain a definable real analytic map

$$\delta : (0,1) \to Z.$$

Furthermore, $\pi_1 \circ \delta$ is semialgebraic and $\pi_3 \circ \delta$ is non-constant.

It follows from

$$\tau_0 = B^t[\tau] \iff \tau = B^{-t}[\tau_0],$$

and $A\Omega_{\tau_0} = \Omega_\tau B$ that first $\tau \circ \delta$ and then $A \circ \delta$ are semialgebraic as well. Here and in the following, we use variables like τ and A also for the corresponding coordinate functions on Z. We can then use that

$$A^{-1}\Omega_\tau x = \Omega_{\tau_0} B^{-1} x = \frac{1}{M}\left(\frac{1}{\det B}\Omega_{\tau_0} R + A_1\tilde{\gamma}_1 + \cdots + A_r\tilde{\gamma}_r\right) + \Omega_{\tau_0} H y$$

to deduce that

$$\Omega_\tau x = \frac{1}{M} A\left(\frac{1}{\det B}\Omega_{\tau_0} R + A_1\tilde{\gamma}_1 + \cdots + A_r\tilde{\gamma}_r\right) + A\Omega_{\tau_0} H y.$$

Now $H \circ \delta$ is semialgebraic. For each $t \in (0,1)$, the rank of $\Omega_{\tau_0}(H \circ \delta)(t)$ is at most $g - k$ by the definition of Z. Note that the first summand in the above expression for $\Omega_\tau x$ is semialgebraic when composed with δ. Therefore we can conclude that the transcendence degree over $\mathbb{C}$ of the complex coordinate functions of the real analytic map

$$\alpha = \psi \circ \pi_3 \circ \delta : (0,1) \to X$$

is at most $g - k + 1$, where $\psi(\tau, x) = (\tau, \Omega_\tau x)$. Since $\pi_3 \circ \delta$ is non-constant, so is α. $\square$

Since α is non-constant, we can choose some $t \in (0,1)$, where the derivative of α does not vanish. Since the Taylor series of α in t must have positive radius of convergence, we can find some holomorphic map $\tilde{\alpha} : D \to X$ from a small open disk D to X such that $t \in D$ and $\tilde{\alpha}|_{D\cap(0,1)} = \alpha|_{D\cap(0,1)}$. By the identity theorem for holomorphic functions, it follows that the transcendence degree over $\mathbb{C}$ of the coordinate functions of $\tilde{\alpha}$ is at most $g - k + 1$ as well.

As the derivative of α does not vanish at t, the map $\tilde{\alpha}$ is non-constant as well. Since every complex analytic irreducible component of X has complex dimension 1, it follows by analytic continuation of the algebraic relations between the coordinate functions of $\tilde{\alpha}$ along the corresponding complex analytic irreducible component of $\exp^{-1}(\mathcal{C}(\mathbb{C}))$ that the Zariski closure of this complex analytic irreducible component of $\exp^{-1}(\mathcal{C}(\mathbb{C}))$ inside $\mathrm{M}_g(\mathbb{C}) \times \mathbb{C}^g$ has dimension at most $g - k + 1$. Theorem 3.1.2 now follows from Theorem 3.7.1. $\square$

3.8.2. Proof of Theorem 3.1.3. If $\pi(\mathcal{C}) = S$, we can apply Theorem 3.1.2 with $k = g$. If $\pi(\mathcal{C}) \neq S$, there exists $s \in S$ such that $\mathcal{C} \subset \mathcal{A}_s$, which means we can apply the non-relative Mordell-Lang conjecture, which Raynaud proved in this case in [146] by reducing it to the theorem of Faltings, to conclude that $\mathcal{A}_\Gamma \cap \mathcal{C} = \phi_s(\Gamma) \cap \mathcal{C}$ is finite unless $\mathcal{C}$ is equal to a translate of an abelian subvariety by a point of $\phi_s(\Gamma) \subset \mathcal{A}_\Gamma$. This argument works for any family $\mathcal{A} \to S$, not only for subfamilies of $\mathfrak{A}_{g,(2l,l)} \to A_{g,(2l,l)}$: We need only Lemma 3.2.2 in order to fix the isogeny, and this also holds for any family after maybe enlarging Γ. If $\mathcal{C}$ is then a translate of an abelian subvariety of $\mathcal{A}_s$, it is a translate of that abelian subvariety by any point on $\mathcal{C}$ and hence also by a point in the isogeny orbit of the original Γ. $\square$

3.8.3. Proof of Corollary 3.1.4. Suppose that $C \cap (\Sigma \times \Gamma')$ is infinite. Let S be the smooth locus of $\mathrm{pr}_1(C) \subset A_{g,(2l,l)}$. We can assume without loss of generality that $\dim S = 1$, otherwise pr_1 is constant and we are done. Let $\pi : \mathfrak{A}_{g,(2l,l)} \to A_{g,(2l,l)}$ be the natural morphism as in Section 3.2 and let $\epsilon : A_{g,(2l,l)} \to \mathfrak{A}_{g,(2l,l)}$ be the zero section.

We apply Theorem 3.1.3 to the non-isotrivial abelian scheme $\mathcal{A} = \pi^{-1}(S) \times_S (S \times A)$ over S with $A_0 = B \times A$, Γ equal to the division closure of $\{(p,q) \in B \times A;\, p \text{ torsion},\, q \in \Gamma'\}$, and $\mathcal{C} = (\epsilon(S)) \times_S \mathrm{pr}_1^{-1}(S) \subset \mathcal{A}$.

A point $p \in \mathrm{pr}_1^{-1}(S) \cap (\Sigma \times \Gamma')$ yields a point $q = (\epsilon(\mathrm{pr}_1(p)), p) \in \mathcal{C}$. If $\phi : B \to \left(\mathfrak{A}_{g,(2l,l)}\right)_{\mathrm{pr}_1(p)}$ is an arbitrary isogeny and 0_B denotes the neutral element of B, then q is the image of $(0_B, \mathrm{pr}_2(p)) \in \Gamma$ under the isogeny $(\phi, \mathrm{id}_A) : B \times A \to \left(\mathfrak{A}_{g,(2l,l)}\right)_{\mathrm{pr}_1(p)} \times A = \mathcal{A}_{\mathrm{pr}_1(p)}$, so $q \in \mathcal{A}_\Gamma$. Since $C \cap (\Sigma \times \Gamma')$ is infinite and $C \setminus \mathrm{pr}_1^{-1}(S)$ is finite, the set $\mathcal{C} \cap \mathcal{A}_\Gamma$ is infinite as well.

If ξ denotes the generic point of S, then we have $A \subset \mathcal{A}_\xi^{\overline{\bar{\mathbb{Q}}(S)}/\bar{\mathbb{Q}}}$. Since $\mathcal{C}$ dominates S, it does not satisfy condition (i) of Theorem 3.1.3, so it has to satisfy condition (ii). This implies that the projection of $\mathcal{C}$ to $S \times A$ must be the graph of a constant map $S \to A$. We deduce that pr_2 is constant. $\square$

CHAPTER 4

Unlikely intersections between isogeny orbits and varieties

Objective considerations of contemporary phenomena compel the conclusion that success or failure in competitive activities exhibits no tendency to be commensurate with innate capacity, but that a considerable element of the unpredictable must invariably be taken into account.

G. Orwell, *Politics and the English Language*

4.1. Introduction

Let K be a field of characteristic zero and let S be a smooth and geometrically irreducible curve, defined over K. Let $\pi : \mathcal{A} \to S$ be an abelian scheme of relative dimension g over S, also defined over K. The zero section $S \to \mathcal{A}$ is called ϵ. The morphism $\pi : \mathcal{A} \to S$ is by definition smooth and proper.

Let $\bar{K}$ be a fixed algebraic closure of K. All varieties that we consider will be defined over $\bar{K}$, if not explicitly stated otherwise. We will identify all varieties with their set of closed points over a prescribed algebraic closure of their field of definition. By "irreducible", we will always mean "geometrically irreducible".

If s is any (possibly non-closed) point of S, we use a subscript s to denote fibers over s. We denote the generic point of S by ξ and fix an algebraic closure $\overline{K(S)}$ of $\bar{K}(S)$. As mentioned above, we identify $\mathcal{A}_\xi$ with its closed points over $\overline{K(S)}$ and thus implicitly with its base change to $\overline{K(S)}$. Let $\left(\mathcal{A}_\xi^{\overline{K(S)}/\bar{K}}, \mathrm{Tr}\right)$ denote the $\overline{K(S)}/\bar{K}$-trace of $\mathcal{A}_\xi$, as defined in Chapter VIII, §3 of [81]. The abelian scheme $\mathcal{A}$ is called isotrivial if Tr is surjective.

Let A_0 be a fixed abelian variety of dimension g. We fix a finite set of $\mathbb{Z}$-linearly independent points $\gamma_1, \dots, \gamma_r$ in $A_0(\bar{K})$. The set can also be empty (i.e. $r = 0$). We set

$$\Gamma = \{\gamma \in A_0;\ \exists N \in \mathbb{N}\colon\ N\gamma \in \mathbb{Z}\gamma_1 + \cdots + \mathbb{Z}\gamma_r\},$$

a subgroup of A_0 of finite rank (and every subgroup of A_0 of finite rank is contained in a group of this form).

We define the isogeny orbit of Γ (in the family $\mathcal{A}$) as

$$\mathcal{A}_\Gamma = \{p \in \mathcal{A}_s;\ s \in S,\ \exists\phi : A_0 \to \mathcal{A}_s \text{ isogeny such that } p \in \phi(\Gamma)\}. \quad (4.1.1)$$

This condition is equivalent to the existence of an isogeny $\psi : \mathcal{A}_s \to A_0$ with $\psi(p) \in \Gamma$.

The following is a special case of the main result of this chapter:

THEOREM 4.1.1. *Suppose that K is a number field, that $\mathcal{A} \to S$ is not isotrivial, and that over $\overline{K(S)}$, $\mathcal{A}_\xi$ is isogenous to a power of an elliptic curve. Suppose further that A_0 is isogenous to E_0^g, where E_0 is an elliptic curve with* $\mathrm{End}(E_0) = \mathbb{Z}$.

Let $\mathcal{V} \subset \mathcal{A}$ be an irreducible subvariety. If $\mathcal{A}_\Gamma \cap \mathcal{V}$ is Zariski dense in $\mathcal{V}$, then one of the following two conditions is satisfied:

- *(i) The variety $\mathcal{V}$ is a translate of an abelian subvariety of $\mathcal{A}_s$ by a point of $\mathcal{A}_\Gamma \cap \mathcal{A}_s$ for some $s \in S$.*
- *(ii) Over $\overline{K(S)}$, the variety $\mathcal{V}_\xi$ is a union of translates of abelian subvarieties of $\mathcal{A}_\xi$ by points in $(\mathcal{A}_\xi)_{\mathrm{tors}}$.*

Compared to similar earlier results, a new aspect is that at once $\mathcal{V}$ is allowed to be of arbitrary dimension and Γ of arbitrary rank. So far, results have been obtained only in the cases when $\mathcal{V}$ is a curve (Chapter 3 of this thesis, Gao [51], Lin-Wang [88]) or Γ contains only torsion points (Gao [51], Habegger [68], Pila [133]). See also [8] and [136] for related results.

If one tries to apply the arguments found in the literature to prove Theorem 4.1.1, the main stumbling block one encounters consists of obtaining a bound for the height of a point in $\mathcal{A}_\Gamma \cap \mathcal{V}$ (outside some degenerate locus) that depends polynomially on the degree of the point. This amounts to solving a Mordell-Lang problem in every fiber, but in a uniform way. Since the known height bounds for the Mordell-Lang problem are ineffective, this is a serious obstacle.

We solve this problem in Theorem 4.4.2, applying a generalized Vojta-Rémond inequality in the form of Theorem A.1.1 (Appendix A). The generalized Vojta-Rémond inequality allows one to compare points from different isogenous fibers. The height bound is still ineffective, but the ineffectivity is now uniformly spread out over all fibers instead of occurring in each fiber separately. Once the height bound is obtained, we proceed along well-known tracks and apply the Pila-Zannier strategy, which is described in Zannier's book [193] together with many problems that can be grouped under the umbrella of "unlikely intersections".

Having an upper bound for the height that depends on the degree of the point is rather unusual compared to previous applications of Vojta's inequality and its generalizations. Theorem 1.3 in [7] is another instance of such a bound that is even logarithmic in the degree of the point. Further examples can be found in [67] and [36]. In our situation, we only obtain a polynomial bound, but this is sufficient for the Pila-Zannier strategy.

If $\mathcal{A}$ is a constant abelian scheme over an irreducible projective base variety S of arbitrary dimension, both defined over $\bar{\mathbb{Q}}$, then von Buhren has obtained in [188] a similar height bound as the one we prove, bounding the height of a point p (outside some degenerate locus) in terms of the height of $\pi(p)$. However, the fact that our result deals with varying abelian varieties rules out a direct application of [188] or of Rémond's generalized Vojta inequality in [154]. The naive idea to just consider the image (or pre-image) under an isogeny of $\mathcal{V}_s$ in A_0 for varying $s \in S$ is ruled out since the degree of the resulting subvariety will in general grow as the degree of the isogeny grows, while the method needs a uniform bound on the degree of the subvariety to produce the desired height bound. Therefore, a generalized Vojta-Rémond inequality is required to handle the family case.

The conditions we put on A_0 and $\mathcal{A}$ are necessary to obtain the height bounds in Section 4.4 insofar as they are crucial to obtain a lower bound for a certain intersection number in Lemma 4.4.5. If we assume that A_0 and $\mathcal{A}$ are principally polarized, then two conditions are necessary for an argument like ours to work (see Section 5.5): First, every cycle on A_0 has to be numerically equivalent to a $\mathbb{Q}$-linear combination of intersections of divisors. Second, for a fiber $\mathcal{A}_s$ ($s \in S$) that is isogenous to A_0 we have to be able to choose a polarized isogeny $\phi : \mathcal{A}_s \to A_0$ such that the index of $\phi^{-1} \operatorname{End}^s(A_0)\phi \cap \operatorname{End}^s(\mathcal{A}_s)$ in $\phi^{-1} \operatorname{End}^s(A_0)\phi$ is bounded independently of $s \in S$. Here, $\operatorname{End}^s(A)$ denotes the additive group of endomorphisms of a principally polarized abelian variety A that are fixed by the Rosati involution.

In the setting of the more general version of Theorem 4.1.1 that we will prove, this second condition will actually only be satisfied on each isotypic factor of $\mathcal{A}_s$ (together with the corresponding isotypic factor of A_0) and we need further restrictions to make sure that an effective cycle on $\mathcal{A}_s$ decomposes into a sum of cartesian products of effective cycles on the isotypic factors (up to numerical equivalence).

The required lower bound for the intersection number can also be obtained under other technical restrictions on $\mathcal{A}$ and A_0 (see Section 5.5), but the case of a fibered power of an elliptic scheme and a corresponding power of a fixed elliptic curve without CM seems to be the most natural one to treat. It is not clear to us if and how such a bound could be obtained in full generality. We emphasize that all parts of the proof apart from the bound in Lemma 4.4.5 can be applied with some necessary modifications to an arbitrary abelian scheme $\mathcal{A}$ over a base curve S.

Theorem 4.1.1 is an instance of Conjecture 3.1.1, which is a slightly modified version of Gao's Conjecture 1.2 in [51], which he calls the André-Pink-Zannier conjecture, in the case of a base curve. Conjecture 3.1.1 is also related to a conjecture of Zannier's (see [51], Conjecture 1.4) and follows from Pink's Conjecture 1.6 in [140] (see [51], Section 8). We refer to Section 3.1 for a more detailed discussion and a comparison of this conjecture with the André-Pink-Zannier conjecture. Conjecture 3.1.1 can be regarded as one relative version of the Mordell-Lang conjecture, proven for abelian varieties by Vojta [186], Faltings [44], and Hindry [72], and in its most general form by McQuillan in [110], analogously to the relative Manin-Mumford results proven by Masser and Zannier in e.g. [103].

Obstacles to proving a reasonable analogue of the conjecture for a base variety S of dimension bigger than 1 are on the one hand the already mentioned inequality between intersection numbers; on the other hand the obstacle that prevented Orr from establishing Theorem 1.2 in [124] beyond the curve case (described on p. 213 of [124]) rears its head as well.

From now on and throughout the rest of this chapter, we suppose that K is a number field and take as $\bar{K} = \bar{\mathbb{Q}}$ its algebraic closure in $\mathbb{C}$. We can now state a slightly more general version of Theorem 4.1.1:

THEOREM 4.1.2. *Suppose that $\mathcal{A} \to S$ is not isotrivial and that over $\overline{\bar{\mathbb{Q}}(S)}$, $\mathcal{A}_\xi$ is isogenous to a product of elliptic curves. Suppose further that A_0 is isogenous to $E_0^{g-g'} \times E_1 \times \cdots \times E_{g'}$, where $0 \leq g' < g$, the E_i are elliptic curves ($i = 0, \ldots, g'$), and $\mathrm{Hom}(E_i, E_j) = \{0\}$ ($i \neq j$) as well as either $g - g' = 1$ or $\mathrm{End}(E_0) = \mathbb{Z}$. We also suppose that $\mathrm{Hom}\left(\mathcal{A}_\xi^{\overline{\bar{\mathbb{Q}}(S)}/\bar{\mathbb{Q}}}, E_0\right) = \{0\}$.*

If $\mathcal{A}_\Gamma \cap \mathcal{V}$ is Zariski dense in $\mathcal{V}$, then one of the following two conditions is satisfied:

(i) The variety $\mathcal{V}$ is a translate of an abelian subvariety of $\mathcal{A}_s$ by a point of $\mathcal{A}_\Gamma \cap \mathcal{A}_s$ for some $s \in S$.

(ii) Over $\overline{\bar{\mathbb{Q}}(S)}$, the variety $\mathcal{V}_\xi$ is a union of translates of abelian subvarieties of $\mathcal{A}_\xi$ by points in $(\mathcal{A}_\xi)_{\mathrm{tors}} + \mathrm{Tr}\left(\mathcal{A}_\xi^{\overline{\bar{\mathbb{Q}}(S)}/\bar{\mathbb{Q}}}(\bar{\mathbb{Q}})\right)$.

The plan of this chapter is as follows: In Section 4.2, we fix some notation. In Section 4.3, we show that it suffices to prove Conjecture 3.1.1 for $\mathcal{V}$ of a certain non-degenerate type without placing any restrictions on $\mathcal{A}$. In Section 4.4, we apply a generalized Vojta-Rémond inequality (see Appendix A) to deduce a height bound of the necessary form for a sufficiently large subset of $\mathcal{A}_\Gamma \cap \mathcal{V}$ if $\mathcal{V}$ is not degenerate and $\mathcal{A}$ and A_0 are of the form described in Theorem 4.1.2. In Section 4.5, we apply the Pila-Zannier strategy and use the height bound we obtained in Section 4.4. The necessary Ax-Lindemann-Weierstrass statement has been proven by Pila in [133]. In Section 4.6, we put all the pieces together and prove Theorem 4.1.2.

4.2. Preliminaries and notation

Let $n_1, \ldots, n_q \in \mathbb{N}$ and let $V \subset \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_q}$ be a subvariety. For $\alpha = (\alpha_1, \ldots, \alpha_q) \in (\mathbb{N} \cup \{0\})^q$ such that $\alpha_1 + \cdots + \alpha_q = \dim V + 1$, Rémond defines a height $h_\alpha(V)$ in [152]. In particular, if $q = 1$ and $V \subset \mathbb{P}^n$ (with $n = n_1$), there is a height $h(V) = h_{\dim V+1}(V)$, coinciding with the one defined in [131]. Similarly, if $\alpha_1 + \cdots + \alpha_q = \dim V$, we use the degree $d_\alpha(V)$ as defined in [152].

If $V_1, \ldots, V_m$ are the irreducible components of V of maximal dimension, then the height $h_\alpha(V)$ is the sum of the $h_\alpha(V_i)$: This follows from the definition of the resultant form in [151], p. 74. To apply the definition, a number field over which V is defined has to be fixed, but the height is independent from the choice of the number field by Proposition 1.28 in [34]. Furthermore, the height is always non-negative: This can be seen by applying Théorème 4 from [85] several times to bound the contributions at the infinite places from below by the corresponding contributions in the height **h** as defined in [129] and then using Proposition 1.12(v) from [129].

4.3. Reduction to the non-degenerate case

In this section, we place no restrictions on $\mathcal{A} \to S$ except that S should be a smooth irreducible curve. We keep our standing assumption that K is a number field, although the results and proofs in this section are valid for arbitrary K of characteristic 0. We will show that it suffices to prove Conjecture 3.1.1 for a certain special type of $\mathcal{V}$ that one might call "non-degenerate". In the proof of Proposition 4.3.2, where this reduction is achieved, we will need to apply the conjecture to another abelian scheme and another abelian variety than the ones we started with. However, the new abelian scheme and abelian variety are obtained by a finite set of operations which preserve many properties of the abelian scheme. We formalize this process in the following definition.

DEFINITION 4.3.1. *A non-empty set $\mathfrak{S}$ of isomorphism classes of pairs ($\mathcal{A} \to S$, A_0) of abelian schemes $\mathcal{A} \to S$, defined over $\bar{\mathbb{Q}}$, with S smooth and irreducible and $\dim S = 1$ and abelian varieties A_0 over $\bar{\mathbb{Q}}$ that are isogenous to infinitely many fibers of $\mathcal{A}$ is called stable if it has the following properties:*

(i) If S' is smooth, irreducible, and defined over $\bar{\mathbb{Q}}$, $\dim S' = 1$, the isomorphism class of $(\mathcal{A} \to S, A_0)$ is in $\mathfrak{S}$, and there is a quasi-finite morphism $S' \to S$ of algebraic curves over $\bar{\mathbb{Q}}$, then the isomorphism class of $(\mathcal{A} \times_S S' \to S', A_0)$ is in $\mathfrak{S}$.

(ii) If the isomorphism class of $(\mathcal{A} \to S, A_0)$ is in $\mathfrak{S}$, $\mathcal{A}' \to S$ is an abelian scheme, defined over $\bar{\mathbb{Q}}$, whose generic fiber is isogenous (over $\bar{\mathbb{Q}}(S)$) to the generic fiber of $\mathcal{A} \to S$, and A_0' is an abelian variety over $\bar{\mathbb{Q}}$ that is isogenous to A_0, then the isomorphism class of $(\mathcal{A}' \to S, A_0')$ is in $\mathfrak{S}$.

Here, two pairs ($\mathcal{A} \to S$, A_0) and ($\mathcal{A}' \to S'$, A_0') are called isomorphic if there exists an isomorphism $S \to S'$ of algebraic curves over $\bar{\mathbb{Q}}$, an isomorphism $\mathcal{A} \to \mathcal{A}' \times_{S'} S$ of abelian schemes over S, and an isomorphism of abelian varieties $A_0 \to A_0'$.

PROPOSITION 4.3.2. *Let $\mathfrak{S}$ be a stable set of isomorphism classes of pairs of abelian schemes and abelian varieties. Suppose that Conjecture 3.1.1 is true for all $(\mathcal{A} \to S, A_0)$ whose class lies in $\mathfrak{S}$ under the additional condition that the union of all translates of positive-dimensional abelian subvarieties of $\mathcal{A}_s$ that are contained in $\mathcal{V}_s$ for some $s \in S$ is not Zariski dense in $\mathcal{V}$. Then it also holds unconditionally for all $(\mathcal{A} \to S, A_0)$ whose class lies in $\mathfrak{S}$.*

We will need the following lemma to prove Proposition 4.3.2. Recall that ξ denotes the generic point of S.

LEMMA 4.3.3. *Suppose that all abelian subvarieties of $\mathcal{A}_\xi$ are defined over $\bar{\mathbb{Q}}(S)$ and that the stabilizer $\mathrm{Stab}(\mathcal{V}_\xi, \mathcal{A}_\xi)$ is finite. Then the union of all translates of positive-dimensional abelian subvarieties of $\mathcal{A}_s$ that are contained in $\mathcal{V}_s$ for some $s \in S$ is not Zariski dense in $\mathcal{V}$.*

Note that this lemma can also be obtained as a consequence of the much more general Theorem 12.2 in [52], at least for $\mathcal{A}$ contained in a suitable universal family and then for arbitrary $\mathcal{A}$ as well. However, we have thought it worthwhile to include a simple proof here that does not make use of the language of mixed Shimura varieties.

PROOF. We first pass to a finite flat cover $S' \to S$ such that S' is smooth and irreducible and every (geometrically) irreducible component of $\mathcal{V}_\xi$ is defined over $\bar{\mathbb{Q}}(S')$. Set $\mathcal{A}' = \mathcal{A} \times_S S'$. Let $\mathcal{V}'$ be an irreducible component of $\mathcal{V} \times_S S' \hookrightarrow \mathcal{A}'$. Since the morphism $\mathcal{V} \times_S S' \to \mathcal{V}$ is flat as the base change of the flat morphism $S' \to S$, we know by Proposition 2.3.4(iii) in [64] that $\mathcal{V}'$ dominates $\mathcal{V}$ and therefore must dominate S'.

If η is the generic point of S' and we identify $\mathcal{A}'_\eta$ with $\mathcal{A}_\xi$ (both being identified with their base change to $\overline{\bar{\mathbb{Q}}(S)}$), then $\mathrm{Stab}(\mathcal{V}'_\eta, \mathcal{A}'_\eta)$ must be finite. Otherwise it would contain a positive-dimensional abelian subvariety of $\mathcal{A}_\xi$, but as all abelian subvarieties of $\mathcal{A}_\xi$ are defined over $\bar{\mathbb{Q}}(S)$, this abelian subvariety would be contained in the stabilizer of $\mathcal{V}_\xi$, which could therefore not be finite. Furthermore, $\mathcal{V}'_\eta = \mathcal{V}' \cap \mathcal{A}'_\eta$ is irreducible by Section 2.1.8 of Chapter 0 of [61] and hence geometrically irreducible by our choice of S'.

If the union of all translates of positive-dimensional abelian subvarieties of $\mathcal{A}_s$ that are contained in $\mathcal{V}_s$ for some $s \in S$ is Zariski dense in $\mathcal{V}$, then the union of all translates of positive-dimensional abelian subvarieties of $\mathcal{A}'_s$ that are contained in $\mathcal{V}'_s$ for some $s \in S'$ is Zariski dense in $\mathcal{V}'$. So we can replace $\mathcal{A}$ and $\mathcal{V}$ by $\mathcal{A}'$ and $\mathcal{V}'$ and assume without loss of generality that $\mathcal{V}_\xi$ is geometrically irreducible.

Let $N \in \mathbb{N}$ be a natural number that is larger than the order of $\mathrm{Stab}(\mathcal{V}_\xi, \mathcal{A}_\xi)$. There are finitely many closed irreducible curves $\mathcal{T}_1, \ldots, \mathcal{T}_R \subset \mathcal{A}$ such that the union of the $\mathcal{T}_i$ ($i = 1, \ldots, R$) is equal to the set of points of exact order N on $\mathcal{A}$: First of all, every irreducible component of the pre-image of $\epsilon(S)$ under the multiplication-by-N morphism $[N]$ dominates S by Proposition 2.3.4(iii) in [64] since $[N]$ is étale, so flat (see [111], Proposition 20.7). Therefore, every irreducible component of $[N]^{-1}(\epsilon(S))$ is of dimension 1. The same holds for any $M \in \mathbb{N}$ that divides N. Furthermore, $[N]^{-1}(\epsilon(S))$ is smooth as $[N]$ is étale and S is smooth. Hence, no two distinct irreducible components of $[N]^{-1}(\epsilon(S))$ intersect each other. So every irreducible component of $[N]^{-1}(\epsilon(S))$ is either contained in $\bigcup_{M|N, M \neq N} [M]^{-1}(\epsilon(S))$ or disjoint from it and our claim follows.

We now consider $\mathcal{W}_i = \mathcal{V} \cap (\mathcal{V} + \mathcal{T}_i)$ for $i \in \{1, \ldots, R\}$. If this variety were equal to $\mathcal{V}$, then we would have $\mathcal{V} \subset \mathcal{V} + \mathcal{T}_i$ and so $\mathcal{V}_\xi \subset \mathcal{V}_\xi + (\mathcal{T}_i)_\xi$. For dimension reasons and thanks to the (geometric) irreducibility of $\mathcal{V}_\xi$, we would get that $\mathcal{V}_\xi = t + \mathcal{V}_\xi$ for a torsion point $t \in \mathcal{A}_\xi$ of order N. This contradicts our choice of N. So $\mathcal{W}_i \subsetneq \mathcal{V}$.

On the other hand, each positive-dimensional abelian variety contains a point of order N, so the union of all translates of positive-dimensional abelian subvarieties of $\mathcal{A}_s$ that are contained in $\mathcal{V}_s$ for some $s \in S$ is contained in $\bigcup_{i=1}^{R} \mathcal{W}_i$. As every $\mathcal{W}_i$ is a proper closed subset of $\mathcal{V}$ and $\mathcal{V}$ is irreducible, the lemma follows. $\square$

Proof. (of Proposition 4.3.2) We can assume without loss of generality that $\pi(\mathcal{V}) = S$ (else the conjecture reduces to the Mordell-Lang conjecture, proven by Faltings, Vojta, and Hindry). After a finite flat base change $S' \to S$ with S' smooth and irreducible and after replacing $\mathcal{A}$ by $\mathcal{A} \times_S S'$ and $\mathcal{V}$ by an irreducible component of $\mathcal{V} \times_S S'$, we can assume that all abelian subvarieties of $\mathcal{A}_\xi$ are defined over $\bar{\mathbb{Q}}(S)$.

Let A' be the irreducible component of $\mathrm{Stab}(\mathcal{V}_\xi, \mathcal{A}_\xi)$ that contains $0_{\mathcal{A}_\xi}$. Then A' is an abelian subvariety of $\mathcal{A}_\xi$. We can now use the Poincaré reducibility theorem to deduce that there exists another abelian subvariety A'' of $\mathcal{A}_\xi$ such that the natural morphism $A' \times A'' \to \mathcal{A}_\xi$ given by restricting the addition morphism $\mathcal{A}_\xi \times \mathcal{A}_\xi \to \mathcal{A}_\xi$ is an isogeny.

The Zariski closures of A' and A'' inside $\mathcal{A}$ are abelian schemes $\mathcal{A}'$ and $\mathcal{A}''$ over S with $A' = \mathcal{A}'_\xi$ and $A'' = \mathcal{A}''_\xi$ by Corollary 6 in Section 7.1 and Proposition 2 in Section 1.4 of [25]. The isogeny between the generic fibers extends to a morphism $\alpha : \mathcal{A}' \times_S \mathcal{A}'' \to \mathcal{A}$, obtained by restricting the addition morphism. We denote the structural morphism $\mathcal{A}' \times_S \mathcal{A}'' \to S$ also by π.

As α is dominant, proper, and maps the image of the zero section to the image of the zero section, it follows that α restricts to an isogeny on each fiber. Let $\mathcal{V}'$

be an irreducible component of $\alpha^{-1}(\mathcal{V})$ that dominates $\mathcal{V}$. Under the hypotheses of Conjecture 3.1.1, the intersection of $\mathcal{V}'$ with the set $(\mathcal{A}' \times_S \mathcal{A}'')_\Gamma$ is Zariski dense in $\mathcal{V}'$, so it suffices to prove the conjecture for $\mathcal{V}'$.

Let $\mathcal{V}''$ be the image of $\mathcal{V}'$ under the projection to $\mathcal{A}''$. Since the projection morphism is proper, $\mathcal{V}''$ is closed in $\mathcal{A}''$. By construction, the generic fiber of $\alpha^{-1}(\mathcal{V})$ contains $\mathcal{A}'_\xi \times \mathcal{V}''_\xi$. It follows that $\mathcal{V}' = \mathcal{A}' \times_S \mathcal{V}''$.

Since A_0 contains only finitely many abelian subvarieties up to automorphism (this is the main result of [86], due to Bertrand for algebraically closed fields of characteristic 0), we can deduce that there exists an abelian subvariety $A_0' \subset A_0$ with the following property: The set of $p = \phi(\gamma) \in \mathcal{V}'$, where $\gamma \in \Gamma$ and $\phi : A_0 \to \mathcal{A}'_{\pi(p)} \times \mathcal{A}''_{\pi(p)}$ is an isogeny, such that there exists an automorphism of A_0 that maps A_0' onto the irreducible component of $\phi^{-1}\left(\mathcal{A}'_{\pi(p)} \times \left\{0_{\mathcal{A}''_{\pi(p)}}\right\}\right)$ containing 0_{A_0} is Zariski dense in $\mathcal{V}'$. Using the Poincaré reducibility theorem over $\bar{\mathbb{Q}}$, we find an abelian subvariety $A_0'' \subset A_0$ such that the natural morphism $A_0' \times A_0'' \to A_0$ is an isogeny.

The pre-image of Γ under this morphism is again a group of finite rank. It is contained in a group $\Gamma' \times \Gamma''$, where Γ', Γ'' are subgroups of finite rank of A_0', A_0'' respectively and $\Gamma' \times \Gamma''$ is stable under $\mathrm{End}(A_0' \times A_0'')$ and equal to its division closure.

Applying Lemma 3.2.2, we find that the intersection of $\mathcal{V}'$ with the set

$$\{(p,q) \in \mathcal{A}'_s \times \mathcal{A}''_s;\ s \in S,\ \exists \phi' : A_0' \to \mathcal{A}'_s, \phi'' : A_0'' \to \mathcal{A}''_s \text{ isogenies such that } (p,q) \in \phi'(\Gamma') \times \phi''(\Gamma'')\}$$

is Zariski dense in $\mathcal{V}'$. But this implies that the intersection of $\mathcal{V}''$ with the set

$$\{p \in \mathcal{A}''_s;\ s \in S,\ \exists \phi : A_0'' \to \mathcal{A}''_s \text{ isogeny such that } p \in \phi(\Gamma'')\}$$

is Zariski dense in $\mathcal{V}''$. Let $\epsilon' : S \to \mathcal{A}'$ be the zero section of $\mathcal{A}'$ and set $\mathcal{V}''' = \epsilon'(S) \times_S \mathcal{V}'' \subset \mathcal{A}' \times_S \mathcal{A}''$.

By Lemma 4.3.3, we now know by hypothesis that Conjecture 3.1.1 is true for $\mathcal{A}' \times_S \mathcal{A}'', \mathcal{V}''', A_0' \times A_0'', \{0_{A_0'}\} \times \Gamma''$ since $\mathrm{Stab}(\mathcal{V}''_\xi, \mathcal{A}''_\xi)$ and hence $\mathrm{Stab}(\mathcal{V}'''_\xi, \mathcal{A}'_\xi \times \mathcal{A}''_\xi)$ must be finite by construction. We obtain that every irreducible component of $\mathcal{V}'''_\xi$ is a translate of an abelian subvariety of $\mathcal{A}'_\xi \times \mathcal{A}''_\xi$ by a point in $(\mathcal{A}'_\xi \times \mathcal{A}''_\xi)_{\mathrm{tors}} + \mathrm{Tr}'\left((\mathcal{A}'_\xi \times \mathcal{A}''_\xi)^{\overline{\bar{\mathbb{Q}}(S)}/\bar{\mathbb{Q}}}(\bar{\mathbb{Q}})\right)$, where $\left((\mathcal{A}'_\xi \times \mathcal{A}''_\xi)^{\overline{\bar{\mathbb{Q}}(S)}/\bar{\mathbb{Q}}}, \mathrm{Tr}'\right)$ denotes the $\overline{\bar{\mathbb{Q}}(S)}/\bar{\mathbb{Q}}$-trace of $\mathcal{A}'_\xi \times \mathcal{A}''_\xi$. As we have $\mathcal{V}''' = \epsilon'(S) \times_S \mathcal{V}''$ and $\mathcal{V}' = \mathcal{A}' \times_S \mathcal{V}''$, we deduce the analogous statement for $\mathcal{V}'$. □

The following lemma will be useful to prove Conjecture 3.1.1 once we have established that the union of all weakly special curves that dominate S and are contained in $\mathcal{V}$ lies Zariski dense in $\mathcal{V}$.

LEMMA 4.3.4. *Suppose that there exists a subgroup $\Gamma' \subset \mathcal{A}_\xi^{\overline{\bar{\mathbb{Q}}(S)}/\bar{\mathbb{Q}}}(\bar{\mathbb{Q}})$ of finite rank such that the irreducible curves $\mathcal{C}$ that are contained in $\mathcal{V}$, dominate S, and satisfy $\mathcal{C}_\xi \subset (\mathcal{A}_\xi)_{\mathrm{tors}} + \mathrm{Tr}(\Gamma')$ lie Zariski dense in $\mathcal{V}$. Then every irreducible component of $\mathcal{V}_\xi$ is a translate of an abelian subvariety of $\mathcal{A}_\xi$ by a point in $(\mathcal{A}_\xi)_{\mathrm{tors}} + \mathrm{Tr}\left(\mathcal{A}_\xi^{\overline{\bar{\mathbb{Q}}(S)}/\bar{\mathbb{Q}}}(\bar{\mathbb{Q}})\right)$.*

Proof. Apply the usual Mordell-Lang conjecture over the field $\overline{\overline{\mathbb{Q}}(S)}$ – which is a consequence of Faltings' main theorem in [44] together with Hindry's Section 6 and Proposition C in [72] – to each irreducible component of $\mathcal{V}_\xi$. $\square$

4.4. Height bounds

In this section, we assume that $A_0 = E_0^{g-g'} \times E_1 \times \cdots \times E_{g'}$ for elliptic curves as in the hypothesis of Theorem 4.1.2. Set $S = Y(2) = \mathbb{A}^1 \backslash \{0, 1\}$ and let

$$\mathcal{E} = \{(\lambda, [x : y : z]) \in Y(2) \times \mathbb{P}^2; y^2 z = x(x-z)(x-\lambda z)\}$$

be the Legendre family of elliptic curves over $Y(2)$. We also assume that $\mathcal{A} = (\mathcal{E} \times_S \cdots \times_S \mathcal{E}) \times_S (E_1 \times \cdots \times E_{g'} \times S)$ and $\pi(\mathcal{V}) = S$. We will show in Section 4.6 that one can always assume this under the hypotheses of Theorem 4.1.2.

There is a canonical open immersion of S into $\mathbb{P}^1$. By choosing a Legendre model for $E_1, \ldots, E_{g'}$, we obtain an immersion of $\mathcal{A}$ into $\mathbb{P}^1 \times (\mathbb{P}^2)^g$. Composing this with first the Veronese embedding of $\mathbb{P}^2$ into $\mathbb{P}^5$ and then the Segre embedding, we obtain an immersion of $\mathcal{A}$ into $\mathbb{P}^1 \times \mathbb{P}^R$ with $R = 6^g - 1$. This induces very ample line bundles $L_{\overline{S}}$ on the compactification $\overline{S} = \mathbb{P}^1$ of S and $\mathcal{L}$ on the associated compactification $\overline{\mathcal{A}}$ of $\mathcal{A}$ such that for each $s \in S$, the line bundle $\mathcal{L}$ restricts to the sixth power of an ample symmetric line bundle that induces a principal polarization on $\mathcal{A}_s$. (We use the Veronese embedding to obtain an even power of an ample line bundle – this will be important in the proof of Lemma 4.4.6.)

By choosing a Legendre model for each of its factors, we can embed A_0 into $(\mathbb{P}^2)^g$ and then as above in $\mathbb{P}^R$ and obtain a symmetric very ample line bundle L_0 on A_0, which is the sixth power of an ample symmetric line bundle that induces a principal polarization on A_0. By applying a linear automorphism on each factor $\mathbb{P}^5$ of $(\mathbb{P}^5)^g$, we can assume that all coordinate hyperplanes intersect A_0 transversally. When embedding $\mathcal{A}$ into $\mathbb{P}^1 \times \mathbb{P}^R$, we can choose the same embeddings into $\mathbb{P}^5$ for $E_1, \ldots, E_{g'}$ as for the corresponding factors of A_0. The embeddings of $\mathcal{E}_s$, E_0, E_1, $\ldots$, $E_{g'}$ into $\mathbb{P}^5$ yield divisors on these curves. As the zero element is mapped to an inflection point of a plane curve in the Legendre model, these divisors are linearly equivalent to six times the zero element of the respective elliptic curve.

We get a (logarithmic projective) height $h_{A_0} = h_{A_0, L_0}$ on A_0. With the usual construction due to Néron and Tate (see [73], Theorem B.5.1) we then obtain a canonical height $\widehat{h}_{A_0}$ on A_0. By our choice of embedding, we have $\widehat{h}_{A_0} = \sum_{i=1}^{g-g'} \widehat{h}_{E_0} \circ \pi_i + \sum_{i=1}^{g'} \widehat{h}_{E_i} \circ \pi_{g-g'+i}$, where $\widehat{h}_{E_j}$ denotes the canonical height on E_j associated to its embedding in $\mathbb{P}^5$ $(j = 0, \ldots, g')$ and π_k denotes the projection onto the k-th factor $(k = 1, \ldots, g)$.

After maybe enlarging the number field K, we can assume that $\mathcal{A}$ (with its structure as an abelian scheme), $\mathcal{L}$, A_0 (with its structure as an abelian variety), L_0 as well as $\mathcal{V}$ and the chosen immersions are all defined over K, $\gamma_1, \ldots, \gamma_r \in A_0(K)$, and every endomorphism of A_0 is defined over K. Since the endomorphism ring of A_0 is finitely generated as a $\mathbb{Z}$-module, we may assume that Γ is mapped into itself by every endomorphism of A_0 by enlarging Γ if necessary. We will generally assume that $r \geq 1$ for simplicity; one can either ensure this by enlarging Γ and K or one can check that our proof also works *mutatis mutandis* if $r = 0$.

The line bundle $L_{\overline{S}}$ on $\overline{S}$ yields a height $h_{\overline{S}}$ on $\overline{S}$. For each $s \in S$, the restriction of $\mathcal{L}$ to $\mathcal{A}_s$ is a very ample symmetric line bundle $\mathcal{L}_s$, which yields a height h_s, induced by the projective embedding $\mathcal{A}_s \hookrightarrow \mathbb{P}^R$, and a canonical height $\widehat{h}_s$ on $\mathcal{A}_s$. As for $\widehat{h}_{A_0}$, this height decomposes as $\widehat{h}_s = \sum_{i=1}^{g-g'} \widehat{h}_s^0 \circ \pi_i + \sum_{i=1}^{g'} \widehat{h}_{E_i} \circ \pi_{g-g'+i}$, where $\widehat{h}_s^0$ denotes the canonical height on $\mathcal{E}_s$ associated to its embedding into $\mathbb{P}^5$ and – by abuse of language – π_j again denotes the projection onto the j-th factor ($j = 1, \ldots, g$). We will denote by h_s^0 and h_{E_j} the usual, not necessarily canonical heights on $\mathcal{E}_s$ and E_j ($j = 0, \ldots, g'$) induced by the embeddings into $\mathbb{P}^5$.

By Lemma 3.2.2, we can fix an isogeny ϕ_s in the definition of $\mathcal{A}_\Gamma$ for each $s \in S$ such that $\mathcal{A}_s$ and A_0 (or equivalently $\mathcal{E}_s$ and E_0) are isogenous. We take $\phi_s = (\psi_s, \ldots, \psi_s, d_s \cdot \mathrm{id}_{E_1 \times \cdots \times E_{g'}})$, where ψ_s is an isogeny from E_0 to $\mathcal{E}_s$ of minimal degree, i.e. there exists no isogeny of smaller degree between them, and $d_s = [\sqrt{\deg \psi_s}]$. Here and in the following, $[\alpha]$ denotes the largest integer less than or equal to α for any real number α.

By Théorème 1.4 of Gaudron-Rémond in [57], which improves a theorem of Masser-Wüstholz ([100], p. 460, and – non-explicitly – [99], p. 1), we have

$$\deg \psi_s \preceq [K(s) : K], \tag{4.4.1}$$

where we will write $f \preceq g$ for (positive) quantities f and g if there exist constants $c > 0$ and $\kappa > 0$, depending on K, A_0, Γ, $\mathcal{A}$, S, and $\mathcal{V}$ as well as $\mathcal{L}$, $L_{\overline{S}}$, L_0, and the immersions associated to these such that

$$f \leq c \max\{1, g\}^\kappa.$$

Note that $\mathcal{A}_s$ and A_0 are both defined over $K(s)$.

All numbered constants $\tilde{c}_1, \tilde{c}_2, \ldots$ in the following will depend only on the quantities that the implicit constants in $\preceq$ are allowed to depend on.

Lemma 4.4.1. *Let $s \in S$ be such that $\mathcal{A}_s$ and A_0 are isogenous. Then there exists a constant $\tilde{c}_1$ such that*

$$h_{\overline{S}}(s) \leq \tilde{c}_1 \max\{\log[K(s) : K], 1\}.$$

Proof. We denote the (stable) Faltings height of an abelian variety A as defined in [42] by $h_F(A)$ and the j-invariant of an elliptic curve E by $j(E)$.

By Faltings' Lemma 5 in [42], we have

$$h_F(\mathcal{A}_s) \leq h_F(A_0) + \frac{\log \deg \phi_s}{2}. \tag{4.4.2}$$

The Faltings height of a product is the sum of the Faltings heights and we have a bound on the difference between $h_F(\mathcal{E}_s)$ and $\frac{1}{12}h(j(\mathcal{E}_s))$ due to Silverman ([172], Proposition 2.1). Finally, the map $s \mapsto j(\mathcal{E}_s)$ extends to a non-constant morphism from $\overline{S} = \mathbb{P}^1$ to $\mathbb{P}^1$ and so we can bound $h_{\overline{S}}(s)$ from above by some multiple of $\max\{h(j(\mathcal{E}_s)), 1\}$ using standard height estimates.

We deduce that

$$h_{\overline{S}}(s) \leq C \max\{h_F(\mathcal{A}_s), 1\} \tag{4.4.3}$$

for some constant C that depends only on $E_1, \ldots, E_{g'}$. Combining (4.4.1), (4.4.2), and (4.4.3), we obtain that

$$h_{\overline{S}}(s) \leq \tilde{c}_1 \max\{\log[K(s) : K], 1\}$$

for some constant $\tilde{c}_1$. □

THEOREM 4.4.2. *Suppose that* $\pi(\mathcal{V}) = S$. *Let* $s \in S$. *Then* $h_s(p) \preceq [K(s) : K]$ *for every point* $p \in \mathcal{V}_s \cap \mathcal{A}_\Gamma$ *that does not lie in a translate of a positive-dimensional abelian subvariety of* $\mathcal{A}_s$ *contained in* $\mathcal{V}_s$.

The proof of Theorem 4.4.2 will occupy the rest of this section. Thanks to Lemma 4.4.1 and (4.4.1), it suffices to find a bound that is polynomial in $\deg \psi_s$ (or equivalently d_s) and $h_{\overline{S}}(s)$. We can assume without loss of generality that $m := \dim \mathcal{V} \geq 2$, otherwise Theorem 4.4.2 follows from Lemma 4.4.1 and elementary height bounds due to the fact that $\pi|_{\mathcal{V}} : \mathcal{V} \to S$ is quasi-finite.

Let $s_1, \ldots, s_m \in S$, then $\mathcal{A}_{s_1}, \ldots, \mathcal{A}_{s_m}$ are abelian varieties with an embedding into $\mathbb{P}^R$ (by projection to the second factor of $\mathbb{P}^1 \times \mathbb{P}^R$). Assume that A_0 is isogenous to $\mathcal{A}_{s_i}$ for all $i = 1, \ldots, m$. Let $\phi_i = \phi_{s_i} : A_0 \to \mathcal{A}_{s_i}$ be the isogeny chosen above (with $\psi_i = \psi_{s_i}$ and $d_i = d_{s_i}$) and let X_i be an irreducible component of the projection to the second factor of $\mathcal{V}_{s_i} = \mathcal{V} \cap (\{s_i\} \times \mathbb{P}^R) \subset \mathcal{A}_{s_i}$ $(i = 1, \ldots, m)$, where we identify $\mathcal{V}$ and $\mathcal{A}$ with their images in $\mathbb{P}^1 \times \mathbb{P}^R$. It follows from $\dim \mathcal{V} = m$, $\pi(\mathcal{V}) = S$, and the Fiber Dimension Theorem (Corollary 14.116 in [58]) that $\dim X_i = m - 1$.

If H denotes a hyperplane in $\mathbb{P}^R$, H' denotes a hyperplane (i.e. a point) in $\mathbb{P}^1$, $\overline{\mathcal{V}}$ denotes the Zariski closure of $\mathcal{V}$ in $\mathbb{P}^1 \times \mathbb{P}^R$, and the class of a cycle C modulo numerical equivalence is denoted by $[C]$, the degrees of the X_i (as subvarieties of $\mathbb{P}^R$) can be estimated as

$$\deg X_i = [\{s_i\} \times X_i] \cdot [\mathbb{P}^1 \times H]^{m-1} \leq [\overline{\mathcal{V}}] \cdot [H' \times \mathbb{P}^R] \cdot [\mathbb{P}^1 \times H]^{m-1} \tag{4.4.4}$$

since every irreducible component of the intersection of $\overline{\mathcal{V}}$ with the hyperplane $\{s_i\} \times \mathbb{P}^R$ is of dimension $m - 1$, $\{s_i\} \times X_i$ is an irreducible component of this intersection (of multiplicity at least 1), and the other irreducible components of the intersection contribute non-negatively to the intersection product by Proposition 8.2 in [47]. Note that the right-hand side is independent of s_i, so $\deg X_i$ is uniformly bounded.

Let $x_i \in X_i \cap \mathcal{A}_\Gamma$ be arbitrary points such that x_i does not lie in a translate of a positive-dimensional abelian subvariety of $\mathcal{A}_{s_i}$ contained in $\mathcal{V}_{s_i}$. Here and in the following, we identify $\mathcal{V}_{s_i}$ and $\mathcal{A}_{s_i}$ with their images in $\mathbb{P}^R$ so that $X_i \subset \mathcal{V}_{s_i} \subset \mathcal{A}_{s_i} \subset \mathbb{P}^R$ $(i = 1, \ldots, m)$.

We set $\zeta_i = \tilde{\phi}_i(x_i) \in \Gamma$, where $\tilde{\phi}_i = (\tilde{\psi}_i, \ldots, \tilde{\psi}_i, d_i \cdot \mathrm{id}_{E_1 \times \cdots \times E_{g'}})$ and $\tilde{\psi}_i : \mathcal{E}_{s_i} \to E_0$ is the isogeny satisfying $\tilde{\psi}_i \circ \psi_i = (\deg \psi_i) \cdot \mathrm{id}_{E_0}$. We record the sequence of inequalities

$$\begin{aligned} d_i^2 \widehat{h}_{s_i}(x_i) \leq \sum_{j=1}^{g-g'} \widehat{h}_{E_0}(\tilde{\psi}_i(\pi_j(x_i))) + \sum_{j=1}^{g'} d_i^2 \widehat{h}_{E_j}(\pi_{g-g'+j}(x_i)) = \widehat{h}_{A_0}(\zeta_i) \\ \leq (\deg \psi_i) \widehat{h}_{s_i}(x_i) \leq 4 d_i^2 \widehat{h}_{s_i}(x_i), \end{aligned} \tag{4.4.5}$$

which follows from

$$\widehat{h}_{E_0}(\tilde{\psi}_i(\pi_j(x_i))) = (\deg \tilde{\psi}_i) \widehat{h}^0_{s_i}(\pi_j(x_i)) = (\deg \psi_i) \widehat{h}^0_{s_i}(\pi_j(x_i))$$

for $j = 1, \ldots, g - g'$.

Assuming that Theorem 4.4.2 is false, we aim to deduce a contradiction with the generalized Vojta-Rémond inequality in Theorem A.1.1 applied to the x_i and X_i $(i = 1, \ldots, m)$. In the following, we show how to choose the various objects and parameters in the generalized Vojta-Rémond inequality in order to arrive at such a contradiction. For $i \in \{1, \dots, m\}$, we choose the very ample line bundle $\mathcal{L}_i$ on X_i from Theorem A.1.1 to be the restriction of $\mathcal{L}_{s_i}$ to X_i and the associated system of homogeneous coordinates $W^{(i)}$ to be the one induced by the closed embedding $X_i \hookrightarrow \mathbb{P}^R$ (so $N_i = R$). We have $X = X_1 \times \cdots \times X_m$, $x = (x_1, \dots, x_m)$, and $u_0 = m(m-1)$.

We define $\|\gamma\| = \sqrt{\widehat{h}_{A_0}(\gamma)}$ for $\gamma \in A_0$, which extends to a norm on $A_0 \otimes \mathbb{R}$. By fundamental properties of the Néron-Tate height, there exists a constant $c_{A_0} > 0$, depending only on A_0 and its embedding into $\mathbb{P}^R$, such that

$$\left|\widehat{h}_{A_0}(\gamma) - h_{A_0}(\gamma)\right| \leq c_{A_0} \tag{4.4.6}$$

for all $\gamma \in A_0$.

LEMMA 4.4.3. *Let $s \in S$ and $p \in \mathcal{A}_s$. There exists a constant $\tilde{c}_2$, depending only on $\mathcal{A}$ and its quasi-projective immersion, such that*

$$|h_s(p) - \widehat{h}_s(p)| \leq \tilde{c}_2 \max\{h_{\overline{S}}(s), 1\}.$$

PROOF. See [196] and note that $\mathcal{E}_s$ is given (in $\mathbb{P}^2$) by an equation $y^2z = \tilde{x}^3 - A(s)\tilde{x}z^2 - B(s)z^3$ with $A(s) = \frac{1}{3}(s^2 - s + 1)$, $B(s) = \frac{1}{27}(s+1)(s-2)(2s-1)$, and $\tilde{x} = x - \frac{s+1}{3}z$ and that the heights of $A(s)$ and $B(s)$ are bounded by a constant multiple of $\max\{h_{\overline{S}}(s), 1\}$ (independently of s). □

Let F be a non-zero linear form on $\mathbb{P}^1$ that vanishes at s_i. Using the theory of heights of subvarieties of multiprojective spaces (see Section 4.2 and [152], whose notation we use), we can estimate the height of X_i as a subvariety of $\mathbb{P}^R$ thanks to Corollaire 2.4 in [152] as

$$h(X_i) = h_{(0, \dim X_i + 1)}(\{s_i\} \times X_i) \leq h_{(0, \dim \mathcal{V})}((\{s_i\} \times \mathbb{P}^R) \cap \bar{\mathcal{V}}).$$

We apply Rémond's multiprojective version of the arithmetic Bézout theorem for the special case of an intersection with a multiprojective "hypersurface" (Théorème 3.4 in [152]) and obtain

$$h_{(0, \dim \mathcal{V})}((\{s_i\} \times \mathbb{P}^R) \cap \bar{\mathcal{V}}) \leq h_{(1, \dim \mathcal{V})}(\bar{\mathcal{V}}) + d_{(0, \dim \mathcal{V})}(\bar{\mathcal{V}}) h_{\bar{\mathcal{V}}, (0, \dim \mathcal{V})}(F).$$

By applying Corollaire 3.6 and Lemme 3.3 from [152], we may bound $h_{\bar{\mathcal{V}}, (0, \dim \mathcal{V})}(F)$ from above by $h(F) + \frac{\log 2}{2}$ (the height of a form used here is defined as in Paragraph 2.1 of [152]). It is elementary that $h(F)$ is bounded from above linearly in terms of $\max\{h_{\overline{S}}(s_i), 1\}$ (with an absolute constant) and so we obtain that

$$h(X_i) \leq \tilde{c}_3 \max\{h_{\overline{S}}(s_i), 1\}. \tag{4.4.7}$$

We have constants $c_1 = c_2 = \Lambda^{\psi(0)}$ from the generalized Vojta-Rémond inequality in Theorem A.1.1, which can be bounded from above independently of the s_i (in fact, we have $c_1 \preceq 1$ as we will see, when fixing the parameters θ, ω, M, t_1, t_2, δ_1, $\dots$, δ_m).

We assume now that

$$\|\zeta_{i+1}\| \geq 4d_{i+1}\sqrt{c_2}\|\zeta_i\| \quad (i = 1, \dots, m-1) \tag{4.4.8}$$

and that

$$h_{s_i}(x_i) > 8c_1(\tilde{c}_2 \max\{h_{\overline{S}}(s_i), 1\} + c_{A_0}) \quad (i = 1, \dots, m) \tag{4.4.9}$$

as well as

$$\|\zeta_i\|^2 \geq 8d_i^2\Lambda^{2\psi(0)}(Mt_2)^{u_0}(\tilde{c}_3 \max\{h_{\overline{S}}(s_i), 1\} + \delta_i) \quad (i = 1, \ldots, m). \tag{4.4.10}$$

We will deduce a contradiction from this together with the condition that the ζ_i lie in a cone of small angle in $\Gamma \otimes \mathbb{R}$, which will imply an ineffective height bound of the desired form thanks to (4.4.1) and (4.4.5), thereby proving Theorem 4.4.2. Of course, the bound depends on the choice of parameters in the generalized Vojta-Rémond inequality, which we will fix later.

We define recursively $b_m = 1$ and

$$b_{i-1} = \left[\frac{b_i\|\zeta_i\|}{\|\zeta_{i-1}\|}\right] + 1 \geq \sqrt{c_2}b_i d_i \quad (i = 2, \ldots, m), \tag{4.4.11}$$

where the lower bound follows from (4.4.8). We then set $a_i = 4(b_i d_i)^2$. The generalized Vojta-Rémond inequality yields additional constants

$$c_3^{(i)} = \Lambda^{2\psi(0)}(Mt_2)^{u_0}(h(X_i) + \delta_i) \quad (i = 1, \ldots, m)$$

(depending on the parameters θ, ω, M, t_1, t_2, δ_1, $\ldots$, δ_m, which will be chosen later).

It follows from (4.4.11) that

$$a_{i-1} \geq c_2 a_i \quad (i = 2, \ldots, m). \tag{4.4.12}$$

We can estimate

$$h_{s_i}(x_i) \geq \frac{1}{2}(h_{s_i}(x_i) + \tilde{c}_2 \max\{h_{\overline{S}}(s_i), 1\}) \geq \frac{1}{2}\widehat{h}_{s_i}(x_i)$$

thanks to Lemma 4.4.3 and (4.4.9). We know from (4.4.7) that

$$c_3^{(i)} \leq \Lambda^{2\psi(0)}(Mt_2)^{u_0}(\tilde{c}_3 \max\{h_{\overline{S}}(s_i), 1\} + \delta_i).$$

It then follows from (4.4.5) and (4.4.10) that

$$c_3^{(i)} \leq \frac{1}{8d_i^2}\|\zeta_i\|^2 \leq \frac{1}{2}\widehat{h}_{s_i}(x_i) \leq h_{s_i}(x_i) \quad (i = 1, \ldots, m). \tag{4.4.13}$$

Assume now that

$$\langle \zeta_i, \zeta_{i+1} \rangle \geq \left(1 - \frac{1}{32c_1}\right)\|\zeta_i\|\|\zeta_{i+1}\|, \tag{4.4.14}$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product associated to $\|\cdot\|$ $(i = 1, \ldots, m-1)$. This means that the angle between ζ_i and ζ_{i+1} with respect to $\langle \cdot, \cdot \rangle$ is small. Since $\Gamma \otimes \mathbb{R}$ is a finite-dimensional Euclidean vector space with respect to $\| \cdot \|$, this partitions the points $p \in \mathcal{V} \cap \mathcal{A}_\Gamma$ that do not lie in a translate of a positive-dimensional abelian subvariety of $\mathcal{A}_{\pi(p)}$ contained in $\mathcal{V}_{\pi(p)}$ into a certain finite number of sets that depends only on c_1 and Γ. After fixing the parameters in the generalized Vojta-Rémond inequality, we will see that $c_1 \preceq 1$ and hence the number of sets can be bounded from above similarly (and independently of the s_i).

We aim to reach a contradiction. We have

$$0 \leq \frac{b_i}{b_{i+1}} - \frac{\|\zeta_{i+1}\|}{\|\zeta_i\|} \leq 1$$

by construction and

$$\left\| \frac{\zeta_i}{\|\zeta_i\|} - \frac{\zeta_{i+1}}{\|\zeta_{i+1}\|} \right\| \leq (4\sqrt{c_1})^{-1}$$

by our assumption on the angle.

It follows from the triangle inequality for $\|\cdot\|$ that

$$\|b_i\zeta_i - b_{i+1}\zeta_{i+1}\| \leq \left(b_i - \frac{b_{i+1}\|\zeta_{i+1}\|}{\|\zeta_i\|} \right) \|\zeta_i\| + b_{i+1}\|\zeta_{i+1}\| \left\| \frac{\zeta_i}{\|\zeta_i\|} - \frac{\zeta_{i+1}}{\|\zeta_{i+1}\|} \right\|,$$

which implies together with the above inequalities that

$$\|b_i\zeta_i - b_{i+1}\zeta_{i+1}\| \leq b_{i+1}\|\zeta_i\| + \frac{b_{i+1}}{4\sqrt{c_1}}\|\zeta_{i+1}\|.$$

Using that $\|\zeta_{i+1}\| \geq 4\sqrt{c_2}\|\zeta_i\| = 4\sqrt{c_1}\|\zeta_i\|$ by (4.4.8), we can conclude that

$$\|b_i\zeta_i - b_{i+1}\zeta_{i+1}\| \leq \frac{b_{i+1}\|\zeta_{i+1}\|}{2\sqrt{c_1}}.$$

This implies thanks to (4.4.5) that

$$4c_1\widehat{h}_{A_0}(b_i\zeta_i - b_{i+1}\zeta_{i+1}) \leq b_{i+1}^2\widehat{h}_{A_0}(\zeta_{i+1}) \leq 4(d_{i+1}b_{i+1})^2\widehat{h}_{s_{i+1}}(x_{i+1})$$

and thus

$$c_1\widehat{h}_{A_0}(b_i\zeta_i - b_{i+1}\zeta_{i+1}) \leq \frac{1}{4}a_{i+1}\widehat{h}_{s_{i+1}}(x_{i+1}).$$

We have already seen that $\widehat{h}_{s_{i+1}}(x_{i+1}) \leq 2h_{s_{i+1}}(x_{i+1})$ and hence

$$c_1\widehat{h}_{A_0}(b_i\zeta_i - b_{i+1}\zeta_{i+1}) \leq \frac{1}{2}a_{i+1}h_{s_{i+1}}(x_{i+1}).$$

We rewrite the left-hand side using the fact that $\widehat{h}_{A_0}$ satisfies the parallelogram law and get

$$c_1\left(2b_i^2\widehat{h}_{A_0}(\zeta_i) + 2b_{i+1}^2\widehat{h}_{A_0}(\zeta_{i+1}) - \widehat{h}_{A_0}(b_i\zeta_i + b_{i+1}\zeta_{i+1})\right) \leq \frac{1}{2}a_{i+1}h_{s_{i+1}}(x_{i+1}).$$

Using (4.4.5), we deduce that

$$\begin{aligned} c_1\Bigg(&\sum_{j=1}^{g-g'} \left(2b_i^2(\deg\tilde{\psi}_i)\widehat{h}^0_{s_i}(\pi_j(x_i)) + 2b_{i+1}^2(\deg\tilde{\psi}_{i+1})\widehat{h}^0_{s_{i+1}}(\pi_j(x_{i+1}))\right) \\ &+ \sum_{j=1}^{g'} \left(2b_i^2d_i^2\widehat{h}_{E_j}(\pi_{g-g'+j}(x_i)) + 2b_{i+1}^2d_{i+1}^2\widehat{h}_{E_j}(\pi_{g-g'+j}(x_{i+1}))\right) \\ &\qquad\qquad - \widehat{h}_{A_0}(b_i\zeta_i + b_{i+1}\zeta_{i+1})\Bigg) \leq \frac{1}{2}a_{i+1}h_{s_{i+1}}(x_{i+1}). \end{aligned}$$

Recall that by abuse of notation, we use π_j for the projection onto the j-th factor in any fiber of $\mathcal{A} \to S$ $(j = 1, \ldots, g)$.

Using (4.4.6), (4.4.9), and Lemma 4.4.3, we obtain by adding up that

$$\begin{aligned}
c_1 \Bigg(\sum_{j=1}^{g-g'} \Big(2b_1^2 (\deg \tilde{\psi}_1) h^0_{s_1}(\pi_j(x_1)) + 2b_m^2 (\deg \tilde{\psi}_m) h^0_{s_m}(\pi_j(x_m))\Big) \\
+ \sum_{j=1}^{g'} \Big(\frac{a_1}{2} h_{E_j}(\pi_{g-g'+j}(x_1)) + \frac{a_m}{2} h_{E_j}(\pi_{g-g'+j}(x_m))\Big) \\
+ \sum_{i=2}^{m-1} \Bigg(\sum_{j=1}^{g-g'} 4b_i^2 (\deg \tilde{\psi}_i) h^0_{s_i}(\pi_j(x_i)) + \sum_{j=1}^{g'} a_i h_{E_j}(\pi_{g-g'+j}(x_i)) \Bigg) \\
- \sum_{i=1}^{m-1} h_{A_0}\left(b_i\zeta_i + b_{i+1}\zeta_{i+1}\right) \Bigg) \leq \frac{1}{2} \sum_{i=2}^{m} a_i h_{s_i}(x_i) \\
+ \sum_{i=1}^{m} \left(4a_i c_1 (\tilde{c}_2 \max\{h_{\overline{S}}(s_i), 1\} + c_{A_0})\right) < \sum_{i=1}^{m} a_i h_{s_i}(x_i). \quad (4.4.15)
\end{aligned}$$

Note that Lemma 4.4.3 implies that

$$\sum_{j=1}^{g-g'} h^0_{s_i}(\pi_j(x_i)) \leq \sum_{j=1}^{g-g'} \widehat{h}^0_{s_i}(\pi_j(x_i)) + \tilde{c}_2 \max\{h_{\overline{S}}(s_i), 1\}$$

if we take $p = (\pi_1(x_i), \ldots, \pi_{g-g'}(x_i), 0_{E_1}, \ldots, 0_{E_{g'}})$.

Recall that $X = X_1 \times \cdots \times X_m$. We consider the morphism $\Psi : X \to A_0^{m-1}$ given by

$$(y_1, \ldots, y_m) \mapsto (b_1\tilde{\phi}_1(y_1) - b_2\tilde{\phi}_2(y_2), \ldots, b_{m-1}\tilde{\phi}_{m-1}(y_{m-1}) - b_m\tilde{\phi}_m(y_m)).$$

If $p_1, \ldots, p_m$, $q_1, \ldots, q_{m-1}$ are the natural projections on X and A_0^{m-1} respectively, we obtain two line bundles on X, namely

$$\mathcal{N}_a = p_1^* \mathcal{L}_1^{\otimes a_1} \otimes \cdots \otimes p_m^* \mathcal{L}_m^{\otimes a_m}$$

and

$$\mathcal{M} = \Psi^*(q_1^* L_0 \otimes \cdots \otimes q_{m-1}^* L_0).$$

Recall that $\mathcal{L}_i$ is the restriction of $\mathcal{L}_{s_i}$ to X_i and that the system of homogeneous coordinates $W^{(i)}$ for $\mathcal{L}_i$ is the one induced by the closed embedding $X_i \hookrightarrow \mathbb{P}^R$. We identify $W^{(i)}$ with $p_i^* W^{(i)}$ $(i = 1, \ldots, m)$.

We let $\sigma : A_0 \times A_0 \to A_0$ and $\delta : A_0 \times A_0 \to A_0$ be the sum and difference morphism respectively. We have $q_i \circ \Psi = \delta \circ ([b_i], [b_{i+1}]) \circ (\tilde{\phi}_i|_{X_i}, \tilde{\phi}_{i+1}|_{X_{i+1}}) \circ (p_i, p_{i+1})$, where $[b]$ denotes the multiplication-by-b morphism on A_0 for $b \in \mathbb{Z}$. Since L_0 is symmetric, we have by Proposition A.7.3.3 in [73] that

$$\delta^* L_0 \simeq \mathrm{pr}_1^* L_0^{\otimes 2} \otimes \mathrm{pr}_2^* L_0^{\otimes 2} \otimes \sigma^* L_0^{\otimes(-1)},$$

where $\mathrm{pr}_i : A_0 \times A_0 \to A_0$ $(i = 1, 2)$ are the natural projections.

It follows that

$$\mathcal{M} \simeq \mathcal{P} \otimes \left(\tilde{\Psi}^*(q_1^* L_0 \otimes \cdots \otimes q_{m-1}^* L_0)\right)^{\otimes -1},$$

where

$$\mathcal{P} = (\tilde{\phi}_1|_{X_1} \circ p_1)^* \left(L_0^{\otimes 2b_1^2}\right) \otimes \bigotimes_{i=2}^{m-1} (\tilde{\phi}_i|_{X_i} \circ p_i)^* \left(L_0^{\otimes 4b_i^2}\right) \otimes (\tilde{\phi}_m|_{X_m} \circ p_m)^* \left(L_0^{\otimes 2b_m^2}\right)$$

and $\tilde{\Psi}$ is defined by replacing every minus in the definition of Ψ with a plus. By our choice of isogenies, the line bundle $\mathcal{P}$ is isomorphic to the very ample line bundle associated to the composition ι' of the closed embedding $X \hookrightarrow \mathcal{A}_{s_1} \times \cdots \times \mathcal{A}_{s_m} \hookrightarrow (\mathbb{P}^5)^{gm}$ with Veronese embeddings of each $\mathbb{P}^5$ of degrees $\alpha_i b_i^2 d_i^2$ and $\alpha_i b_i^2 \deg \psi_i$ ($i = 1, \ldots, m$), where $\alpha_i = 2$ if $i \in \{1, m\}$ and 4 otherwise, and finally the Segre embedding.

It follows from $\deg \psi_i \leq 4d_i^2$ that $\mathcal{P}$ injects into $\mathcal{N}_a^{\otimes 4}$, so we can set $t_1 = 4$. The embedding ι' yields a system of homogeneous coordinates Ξ for $\mathcal{P}$ and the injection can be chosen such that they map to monomials in the $W^{(i)}$ in $\Gamma(X, \mathcal{N}_a^{\otimes 4})$.

The line bundle $\mathcal{M}$ is nef as it is a tensor product of pull-backs of nef line bundles under proper morphisms and we have

$$\mathcal{P} \otimes \mathcal{M}^{\otimes -1} \simeq \tilde{\Psi}^*(q_1^* L_0 \otimes \cdots \otimes q_{m-1}^* L_0).$$

We will see in Lemma 4.4.8 that the line bundle on the right injects into $\mathcal{N}_a^{\otimes 8}$ (so we choose $t_2 = 8$). In the same lemma, we show that this line bundle is generated by a set of sections Z of cardinality $M = (R+1)^{m-1}$ satisfying

$$h(Z(x)) = \sum_{i=1}^{m-1} h_{A_0}(b_i \zeta_i + b_{i+1} \zeta_{i+1}).$$

Furthermore, these sections correspond under the given injection to polynomials of multidegree $t_2 a$ in the projective coordinates $W^{(i)}$ on X of height bounded by $\sum_{i=1}^m a_i \delta_i$, where each δ_i is bounded polynomially in d_i and $h_{\overline{S}}(s_i)$.

It should be noted here that the mentioned injection is achieved by writing $\mathcal{N}_a^{\otimes t_2}$ as the tensor product of $\mathcal{P} \otimes \mathcal{M}^{\otimes(-1)}$ with a globally generated line bundle and then sending each section to its tensor product with a global section of this second line bundle that does not vanish at x. Thus, the injection depends on x, but the number of choices for the injection can be bounded independently of x. The set Z is always the same but corresponds to different multihomogeneous polynomials depending on the chosen injection. However, the height of the polynomials is bounded independently of the choice of injection. Analogous statements hold for the injection of $\mathcal{P}$ into $\mathcal{N}_a^{\otimes t_1}$ and Ξ.

With $h_{\mathcal{M}}(x) = h(\Xi(x)) - h(Z(x))$, the inequality (4.4.15) then amounts to

$$c_1 h_{\mathcal{M}}(x) < h_{\mathcal{N}_a}(x), \tag{4.4.16}$$

which yields the desired contradiction with Theorem A.1.1 provided that we can choose the parameters in Theorem A.1.1 such that all conditions of Theorem A.1.1 are satisfied and such that the parameters are bounded in the required way. We will achieve this in the following lemmata.

First of all, we set $\mathcal{X} = X$ and $\pi = \mathrm{id}$. Consequently, we have $\mathcal{Y} = Y$ for every subproduct $Y \subset X$ in Theorem A.1.1. We set $\omega = 0$, which will be justified by Lemma 4.4.5. Before stating and proving this lemma (and Lemma 4.4.4 from which it follows), we have to recall some terminology: By a cycle on $\mathcal{A}_s$ ($s \in S$), we mean a formal sum of irreducible subvarieties of $\mathcal{A}_s$ (with integer coefficients). If C is a cycle on $\mathcal{A}_s$, we denote by $[C]$ its equivalence class modulo numerical equivalence. We will also call $[C]$ the class of C for short. An effective class is a class a positive integral multiple of which contains an effective cycle (i.e. a formal sum of irreducible subvarieties with non-negative integer coefficients). For a natural number n and a cycle C, we denote by $n[C]$ the class $[C+C+\cdots+C]$ (n times), where the addition takes place in the additive group of cycles and not on the abelian variety.

LEMMA 4.4.4. *There exists a natural number N, depending only on g, such that for every $s \in S$ such that $\mathcal{E}_s$ is isogenous to E_0 and for every cycle Y on $\mathcal{A}_s$, the class $N[Y]$ lies in the subring of the ring of cycles modulo numerical equivalence that is generated under addition and intersection product by $\mathcal{E}_s^{g-g'} \times E_1 \times \cdots \times E_{g'}$ itself together with the hypersurfaces defined by $\pi_j(z) = 0_{\mathcal{E}_s}$ $(j = 1, \ldots, g-g')$, $\pi_{g-g'+j}(z) = 0_{E_j}$ $(j = 1, \ldots, g')$, and $\pi_j(z) = \pi_k(z)$ $(1 \leq j < k \leq g-g')$.*

In the proof of this lemma, we use the restrictions we placed on $\mathcal{A}$ and A_0 in a crucial way.

PROOF. Let $s \in S$ such that $\mathcal{E}_s$ is isogenous to E_0 and let Y be a cycle on $\mathcal{A}_s$. We assume that Y is equidimensional of dimension $\dim Y$. Let us denote the subring mentioned in the lemma by $\mathfrak{D}$. We denote by $\mathfrak{D}_k$ the intersection of $\mathfrak{D}$ with the additive subgroup of all classes of equidimensional cycles of dimension k. If $\dim Y = g$, the assertion of the lemma is true with $N = 1$.

For the proof, we will use different equivalence relations on the set of cycles on $\mathcal{A}_s$, namely algebraic, homological, and numerical equivalence. For definitions, see Chapters 10 and 19 of [47]. In order to use homological equivalence, we will tacitly identify $\mathcal{A}_s$ with its base change to $\mathbb{C} \supset \bar{\mathbb{Q}}$; if the statement of the lemma holds over $\mathbb{C}$, it automatically holds over $\bar{\mathbb{Q}}$ as well. Note that the Borel-Moore homology used by Fulton coincides with the usual singular homology since the ambient variety $\mathcal{A}_s$ is projective. Also, since we are working on an abelian variety, numerical and homological equivalence coincide (see [87]) and for codimension 1 cycles all three equivalence relations coincide (see [47], Section 19.3.1, and use the fact that the Néron-Severi group of an abelian variety is torsion-free).

There is a natural isomorphism between the additive group of cycles of codimension 1 modulo algebraic equivalence and the Néron-Severi group of $\mathcal{A}_s$. We fix ample line bundles on $\mathcal{E}_s$ and $E_1, \ldots, E_{g'}$ that induce principal polarizations on these elliptic curves. The tensor product of the pull-backs of these line bundles under the projections on the factors induces a principal polarization on $\mathcal{A}_s$. This polarization induces an isomorphism between the Néron-Severi group of $\mathcal{A}_s$ and the additive group of endomorphisms of $\mathcal{A}_s$ that are fixed by the corresponding Rosati involution, after tensoring both with $\mathbb{Q}$ (see [111], Proposition 17.2). Since the polarization is principal, the proof of Proposition 17.2 in [111] shows that we also get an isomorphism without tensoring with $\mathbb{Q}$.

By our hypotheses on $\mathrm{Hom}(E_i, E_j)$ $(i, j = 0, \ldots, g')$, the endomorphism ring of $\mathcal{A}_s$ is naturally isomorphic to either $\mathrm{M}_{g-g'}(\mathbb{Z}) \times E$ or $\mathrm{End}(\mathcal{E}_s) \times E$, where $E = \prod_{i=1}^{g'} \mathrm{End}(E_i)$. In both cases, the set of endomorphisms that are fixed by the Rosati involution corresponds to $M^s_{g-g'}(\mathbb{Z}) \times \mathbb{Z}^{g'}$, where $M^s_{g-g'}(\mathbb{Z})$ denotes the set of symmetric $(g-g') \times (g-g')$-matrices with entries in $\mathbb{Z}$.

We fix a $\mathbb{Z}$-basis for this additive group by choosing the standard basis for $\mathbb{Z}^{g'}$ and the basis $\{A_i, A_{j,k}; i = 1, \ldots, g-g', 1 \leq j < k \leq g-g'\}$ for $M^s_{g-g'}(\mathbb{Z})$ with $A_i = (a^i_{r,s})_{1 \leq r,s \leq g-g'}$, $A_{j,k} = (a^{j,k}_{t,u})_{1 \leq t,u \leq g-g'}$, $a^i_{r,s} = 1$ if $r = s = i$ and 0 otherwise, and $a^{j,k}_{t,u} = 1$ if $t = u \in \{j,k\}$, -1 if $(t,u) \in \{(j,k),(k,j)\}$, and 0 otherwise.

For a line bundle L on $\mathcal{A}_s$, we denote the associated homomorphism from $\mathcal{A}_s$ to $\widehat{\mathcal{A}}_s$ by $\phi_L : \mathcal{A}_s \to \widehat{\mathcal{A}}_s$. If $\widetilde{\mathcal{L}_s}$ is the ample line bundle that induces the principal polarization on $\mathcal{A}_s$ and $\chi \in \mathrm{End}(\mathcal{A}_s)$ is fixed by the corresponding Rosati involution, then we have $\phi_{\chi^*\widetilde{\mathcal{L}_s}} = \phi_{\widetilde{\mathcal{L}_s}} \circ \chi \circ \chi$. Using this fact, we can compute that under the above isomorphism the chosen basis of $M^s_{g-g'}(\mathbb{Z}) \times \mathbb{Z}^{g'}$ corresponds precisely to the collection of hypersurfaces that generate $\mathfrak{D}$ (modulo algebraic equivalence). This proves the assertion of the lemma in codimension 1 with $N = 1$.

Thanks to Murty, who showed in [121] that (after tensoring with $\mathbb{Q}$) any cycle on a product of elliptic curves over $\mathbb{C}$ is homologically (and hence numerically) equivalent to a $\mathbb{Q}$-linear combination of intersections of divisors, we then know that $[Y]$ lies in $\mathbb{Q}\mathfrak{D}_{\dim Y} = \mathfrak{D}_{\dim Y} \otimes_{\mathbb{Z}} \mathbb{Q}$. The intersection product yields (by definition of numerical equivalence) a non-degenerate bilinear form $\mathbb{Q}\mathfrak{D}_{\dim Y} \times \mathbb{Q}\mathfrak{D}_{g-\dim Y} \to \mathbb{Q}$, which we will denote by $\langle \cdot, \cdot \rangle$. In particular, we have $d := \dim \mathbb{Q}\mathfrak{D}_{\dim Y} = \dim \mathbb{Q}\mathfrak{D}_{g-\dim Y}$.

We choose a $\mathbb{Q}$-basis $(v_1, \ldots, v_d)$ for $\mathbb{Q}\mathfrak{D}_{\dim Y}$ and a $\mathbb{Q}$-basis $(w_1, \ldots, w_d)$ for $\mathbb{Q}\mathfrak{D}_{g-\dim Y}$, both consisting of intersections of the hypersurfaces described in the lemma. Note that the intersection product of v_i and w_j is either 0 or 1 for all i and j since any collection of g hypersurfaces as in the lemma either meets in a positive-dimensional component (so does not meet at all after translating one of the hypersurfaces by a sufficiently generic point) or meets transversely in the origin of $\mathcal{A}_s$.

We can write $[Y] = \sum_{i=1}^d \lambda_i v_i$ with $\lambda_i \in \mathbb{Q}$ $(i = 1, \ldots, d)$. The intersection product of Y with each of the w_j is an integer, so we get that $\sum_{i=1}^d \langle v_i, w_j \rangle \lambda_i \in \mathbb{Z}$ $(j = 1, \ldots, d)$. We conclude that $\Delta\lambda_i \in \mathbb{Z}$ $(i = 1, \ldots, d)$, where Δ is the (non-zero) determinant of the matrix $(\langle v_i, w_j \rangle)_{i,j=1,\ldots,d}$. Now d can be bounded in terms of g and then $|\Delta|$ can be bounded in terms of g by Hadamard's determinant inequality – note that each entry of the matrix is either 0 or 1. By taking the least common multiple of all possible Δ, we obtain a natural number N such that $N[Y] \in \mathcal{D}_{\dim Y}$ and N depends only on g. We can now take N to be the least common multiple of all the N we obtain for varying $\dim Y \in \{0, 1, \ldots, g\}$ and the lemma follows. $\square$

LEMMA 4.4.5. *Suppose that x_i does not lie in a translate of a positive-dimensional abelian subvariety of $\mathcal{A}_{s_i}$ that is contained in X_i $(i = 1, \ldots, m)$. There exists an integer $\theta \geq 1$, depending only on g and m, but independent of x, X, and Y, such that*

$$\mathcal{M}^{\cdot \dim(Y)} \cdot Y \geq \theta^{-1} \prod_{i=1}^{m} a_i^{\dim(Y_i)}$$

for every subproduct $Y = Y_1 \times \cdots \times Y_m \subset X$ such that $Y_i \subset X_i$ is an irreducible subvariety $(i = 1, \ldots, m)$ and $x \in Y$.

PROOF. We define a finite morphism $\Phi : X \to A_0^m$ by

$$\Phi(y_1, \ldots, y_m) = (\tilde{\phi}_1(y_1), \ldots, \tilde{\phi}_m(y_m)).$$

The morphism Ψ factorizes as $\Psi' \circ \Phi$ with

$$\Psi'(y_1, \ldots, y_m) = (b_1 y_1 - b_2 y_2, \ldots, b_{m-1} y_{m-1} - b_m y_m)$$

and we get a line bundle $\tilde{\mathcal{M}} = \Psi'^*(q_1^* L_0 \otimes \cdots \otimes q_{m-1}^* L_0)$ on A_0^m such that $\mathcal{M} = \Phi^* \tilde{\mathcal{M}}$.

It follows from the projection formula that

$$\mathcal{M}^{\cdot \dim(Y)} \cdot [Y] = \tilde{\mathcal{M}}^{\cdot \dim(Y)} \cdot \Phi_*([Y]). \tag{4.4.17}$$

By a crucial homogeneity result of Faltings (see [43], Lemma 4.2, or [187], Corollary 11.4), we have

$$\tilde{\mathcal{M}}^{\cdot \dim(\Phi(Y))} \cdot \Phi_*([Y]) = \left(\prod_{i=1}^{m} b_i^{2 \dim Y_i} \right) (\mathcal{M}_1^{\cdot \dim(\Phi(Y))} \cdot \Phi_*([Y])), \tag{4.4.18}$$

where $\mathcal{M}_1 = \Psi_1^*(q_1^* L_0 \otimes \cdots \otimes q_{m-1}^* L_0)$ and

$$\Psi_1(y_1, \ldots, y_m) = (y_1 - y_2, \ldots, y_{m-1} - y_m).$$

We will now show that

$$\mathcal{M}_1^{\cdot \dim(\Phi(Y))} \cdot \Phi_*([Y]) \geq \tilde{\theta}^{-1} \prod_{i=1}^{m} d_i^{2 \dim Y_i}$$

for some integer $\tilde{\theta} \geq 1$, depending only on g and m.

Let N be the natural number furnished by Lemma 4.4.4 and let $i \in \{1, \ldots, m\}$. The cycle NY_i on $\mathcal{A}_{s_i}$ is numerically equivalent to a sum $\sum_{I \subset \{1, \ldots, g'\}} C_I \times C'_I$ by Lemma 4.4.4, where C_I is a $\mathbb{Z}$-linear combination of cycles on $\mathcal{E}_{s_i}^{g-g'}$, each given by a collection of equations of the form $\pi_j(z) = 0_{\mathcal{E}_{s_i}}$ or $\pi_j(z) = \pi_k(z)$ $(j \neq k)$, and $C'_I = \{(z_1, \ldots, z_{g'}) \in E_1 \times \cdots \times E_{g'}; z_j = 0_{E_j} \forall j \in I\}$. Furthermore, we can assume that C_I is either zero or equidimensional of dimension $\dim Y_i - \dim C'_I \geq 0$. It follows that

$$[C_I] = N(\pi_1, \ldots, \pi_{g-g'})_* \left([Y_i] \cdot \left[\mathcal{E}_{s_i}^{g-g'} \times C'_{\{1,\ldots,g'\} \setminus I} \right] \right).$$

On an abelian variety, the intersection of two effective classes is again an effective class since C and $u + C$ are algebraically equivalent for any element u of the abelian variety and any irreducible subvariety C; if C and D are two irreducible subvarieties, then $u + C$ will intersect D dimensionally transversely for all u in an open Zariski dense set by the Fiber Dimension Theorem (Corollary 14.116 in [58]). Furthermore, the push-forward of an effective class is always an effective class. So $[C_I]$ and hence $[C_I \times C'_I]$ is an effective class.

Here and in the following we implicitly use that the cartesian product with a numerically trivial cycle stays numerically trivial – this follows from the fact that the class of a cartesian product is the intersection product of the pull-backs of the classes of its factors and the fact that numerical equivalence is preserved under pull-back with respect to a flat morphism between non-singular complete varieties (see Example 19.1.6 in [47]).

Using the special form of C_I and C'_I, we compute that if C_I is non-zero, then $\left(\tilde{\phi}_i\right)_* ([C_I \times C'_I]) = (\deg \tilde{\psi}_i)^{\dim C_I} d_i^{2 \dim C'_I} [E_I]$, where E_I is a cycle on A_0 and its class $[E_I]$ is effective. If C_I is zero, we set E_I to zero as well. It follows from the definition of $d_i = d_{s_i}$ in the paragraph before (4.4.1) that $d_i^2 \leq \deg \tilde{\psi}_i \leq 4d_i^2$. Hence, we deduce that

$$4^{\dim Y_i} d_i^{2\dim Y_i}[Z_i] \geq N\left(\tilde{\phi}_i\right)_*([Y_i]) \geq d_i^{2\dim Y_i}[Z_i],$$

where the class of

$$Z_i = \sum_{I \subset \{1,\ldots,g'\}} E_I$$

is effective and we write $[U] \geq [V]$ if $[U] - [V]$ is an effective class.

It follows that

$$\mathcal{M}_1^{\cdot \dim(\Phi(Y))} \cdot \Phi_*([Y]) \geq \frac{1}{N^m}\left(\prod_{i=1}^m d_i^{2\dim Y_i}\right)\mathcal{M}_1^{\cdot \dim(\Phi(Y))} \cdot [Z_1 \times \cdots \times Z_m]. \quad (4.4.19)$$

The right-hand side here is positive since

$$\mathcal{M}_1^{\cdot \dim(\Phi(Y))} \cdot \Phi_*([Y]) \leq \frac{1}{N^m}\left(\prod_{i=1}^m (4d_i^2)^{\dim Y_i}\right)\mathcal{M}_1^{\cdot \dim(\Phi(Y))} \cdot [Z_1 \times \cdots \times Z_m]$$

and

$$\mathcal{M}_1^{\cdot \dim(\Phi(Y))} \cdot \Phi_*([Y]) > 0.$$

This last inequality follows from the fact that the morphism Ψ_1 restricted to $\Phi(Y)$ is generically finite, which can be shown by adapting the proof of Lemme 2.1 in [150]. We simply have to note that $\tilde{\phi}_i(Y_i)$ cannot have a positive-dimensional stabilizer since otherwise the same would hold for Y_i and then x_i would lie in a translate of a positive-dimensional abelian subvariety that is contained in X_i, which contradicts our assumption.

As a positive integer is greater than or equal to 1, we obtain that

$$\mathcal{M}_1^{\cdot \dim(\Phi(Y))} \cdot [Z_1 \times \cdots \times Z_m] \geq 1$$

and therefore

$$\mathcal{M}_1^{\cdot \dim(\Phi(Y))} \cdot \Phi_*([Y]) \geq \frac{1}{N^m}\left(\prod_{i=1}^m d_i^{2\dim Y_i}\right)$$

by (4.4.19).

Since $a_i = 4(b_i d_i)^2$ $(i = 1, \ldots, m)$, the lemma now follows from combining this inequality with (4.4.17) and (4.4.18). $\square$

LEMMA 4.4.6. *Let $s \in S$ be such that $\mathcal{E}_s$ and E_0 are isogenous and let $\psi : \mathcal{E}_s \to E_0$ be an isogeny. Recall that we have embedded $\mathcal{E}_s$ and E_0 into $\mathbb{P}^5$. There exists a constant $\tilde{c}$, depending only on A_0, $\mathcal{A}$, and their (quasi-)projective immersions, such that ψ is given by 6 homogeneous polynomials of degree $\deg\psi$ in the coordinates on $\mathbb{P}^5$ and the height of the set of coefficients of all these polynomials is at most*

$$\tilde{c}(\deg\psi)^{10}\max\{h_{\overline{S}}(s), 1\}.$$

PROOF. The embeddings into $\mathbb{P}^5$ yield line bundles $\mathcal{L}'_s$ on $\mathcal{E}_s$ and L'_0 on E_0. Recall that they are both sixth tensor powers of the line bundle associated to the divisor given by the zero element of the respective elliptic curve. It follows that $\psi^* L'_0$ corresponds to six times the divisor associated to the kernel of ψ. This divisor is linearly equivalent to $6(\deg\psi)0_{\mathcal{E}_s}$. It follows that $\psi^* L'_0 \simeq \mathcal{L}_s'^{\otimes \deg\psi}$.

The coordinates on $\mathbb{P}^5$ pull back under ψ to global sections of this line bundle and our first goal is to show that these sections can in fact be written as homogeneous polynomials of degree $\deg\psi$ in the coordinates on $\mathcal{E}_s \subset \mathbb{P}^5$. Alternatively, we aim to show that the divisor given by the pull-back of a coordinate hyperplane under ψ is cut out by a single homogeneous form of degree $\deg\psi$. Since $H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(k)) = \{0\}$ for all integers k by Theorem III.5.1(b) in [71], the embedding of $\mathcal{E}_s$ into $\mathbb{P}^2$ is projectively normal. Therefore, the same holds for the embedding of $\mathcal{E}_s$ into $\mathbb{P}^5$ and the desired claim follows.

Thus, the isogeny ψ is given by six homogeneous polynomials $P_0, \ldots, P_5$ of degree $d = \deg\psi$ in six variables. We know that ψ maps the torsion points of $\mathcal{E}_s$ onto the torsion points of E_0. The equalities $\psi(p_i) = q_i$ $(i = 1, 2, \ldots)$, where the $p_i = [p_{i,0} : \cdots : p_{i,5}]$ are the torsion points of $\mathcal{E}_s \subset \mathbb{P}^5$ in some order and the $q_i = [q_{i,0} : \cdots : q_{i,5}]$ are torsion points of $E_0 \subset \mathbb{P}^5$, give rise to homogeneous linear equations of the form

$$q_{i,k_1} P_{k_2}(p_{i,0}, \ldots, p_{i,5}) - q_{i,k_2} P_{k_1}(p_{i,0}, \ldots, p_{i,5}) = 0 \quad (0 \leq k_1 < k_2 \leq 5)$$

that the coefficients of the P_k $(k = 0, \ldots, 5)$ satisfy.

Let $D = 6\binom{d+5}{5}$ be the number of coefficients of all the P_k $(k = 0, \ldots, 5)$ and let $\rho \leq D$ be the rank of the (infinite) system of linear equations. We may choose ρ of these equations such that the resulting matrix has rank ρ. Consequently, there is a minor of dimension ρ with non-vanishing determinant. Using Cramer's rule, we obtain a basis for the space of solutions by taking determinants of suitable $\rho \times \rho$-matrices.

The entries of such a matrix are either 0 or have the form

$$\pm q_{i,k} p_{i,0}^{j_0} \cdots p_{i,5}^{j_5},$$

where $j_0 + \cdots + j_5 = d$. If Δ is its determinant and v is a normalized valuation of a number field F containing all coefficients of the linear equations, we can estimate

$$|\Delta|_v \leq \epsilon(v) \prod_{l=1}^{\rho} \left(\max_k\{|q_{i_l,k}|_v\} \max_n\{|p_{i_l,n}|_v\}^d \right),$$

where $\epsilon(v)$ equals 1 or $\rho!$ according to whether v is finite or infinite and $i_1, \ldots, i_\rho$ are the indices occurring in the rows of the matrix. For a basis element $z = [\Delta_1 : \cdots : \Delta_D]$, where each Δ_k is either a determinant as above or zero, but not all Δ_k are zero, we obtain that

$$h(z) \leq \log(\rho!) + \rho \max_l\{h_{E_0}(q_{i_l})\} + d\rho \max_l\{h_s^0(p_{i_l})\}.$$

Using $\rho \leq D$ and $\rho! \leq \rho^\rho$, we further deduce that

$$h(z) \leq D \log D + D \max_l\{h_{E_0}(q_{i_l})\} + dD \max_l\{h_s^0(p_{i_l})\}.$$

Since $\widehat{h}_{E_0}(q_{i_l}) = \widehat{h}_s^0(p_{i_l}) = 0$, we obtain that $h_{E_0}(q_{i_l}) \leq c_{A_0}$ and $h_s^0(p_{i_l}) \leq \tilde{c}_2 \max\{h_{\overline{S}}(s), 1\}$ from (4.4.6) and Lemma 4.4.3 (for all l). Together with the above this implies that

$$h(z) \leq \tilde{c} D^2 \max\{1, h_{\overline{S}}(s)\}$$

for some constant $\tilde{c}$ that depends only on A_0, $\mathcal{A}$, and their (quasi-)projective immersions. Since $D \leq 36d^5$, the lemma follows as soon as we have shown that we can choose an element of the basis which indeed represents ψ.

Now by construction, each element of this basis corresponds to a sextuple of polynomials $(P_0, \ldots, P_5)$ such that for each torsion point $p \in \mathcal{E}_s$ we have either $P_0(p) = \cdots = P_5(p) = 0$ or $[P_0(p) : \cdots : P_5(p)] = \psi(p)$. After discarding those basis elements that vanish everywhere, we can assume that each basis element represents ψ on an open Zariski dense subset of $\mathcal{E}_s$. Here, we use that the torsion points lie Zariski dense in $\mathcal{E}_s$. We choose one such element $(P_0, \ldots, P_5)$. Note that we have not discarded everything since we already know that there exists some sextuple of polynomials that represents ψ.

If H_k is the scheme-theoretic intersection of E_0 with the coordinate hyperplane in $\mathbb{P}^5$ given by the vanishing of the k-th coordinate $(k = 0, \ldots, 5)$, then P_k certainly vanishes on $\psi^{-1}(H_k)$. Recall that the coordinate hyperplanes of $\mathbb{P}^5$ intersect E_0 transversely, so H_k is reduced and $\psi^{-1}(H_k) \neq \mathcal{E}_s$. If this is the precise zero locus of each P_k, then we are done. Otherwise, the zero loci of all P_k share an irreducible component of codimension 1.

We identify H_k with the divisor on E_0 that is associated to it. Each of the P_k defines an effective divisor D_k on $\mathcal{E}_s$ that is linearly equivalent to $\psi^*(H_k)$ (being a homogeneous polynomial of degree d that does not vanish identically on $\mathcal{E}_s$ since $\psi^{-1}(H_k) \neq \mathcal{E}_s$). At the same time, the support of D_k contains the support of $\psi^*(H_k)$. Since H_k is reduced and ψ is unramified, it follows that $\psi^*(H_k)$ is reduced and hence $D_k - \psi^*(H_k)$ is effective. If this difference does not vanish for some k, this would give us a non-trivial effective divisor that is linearly equivalent to 0, a contradiction. Hence, the polynomials $(P_0, \ldots, P_5)$ define ψ everywhere on $\mathcal{E}_s$. $\square$

LEMMA 4.4.7. *Let $s', s'' \in S$ and let $b', b'' \in \mathbb{N}$. Let $\tilde{\psi}_{s'} : \mathcal{E}_{s'} \to E_0$ and $\tilde{\psi}_{s''} : \mathcal{E}_{s''} \to E_0$ be isogenies and set $\tilde{\phi}_{s'} = (\tilde{\psi}_{s'}, \ldots, \tilde{\psi}_{s'}, d' \cdot \mathrm{id}_{E_1 \times \cdots \times E_{g'}}) : \mathcal{A}_{s'} \to A_0$, $\tilde{\phi}_{s''} = (\tilde{\psi}_{s''}, \ldots, \tilde{\psi}_{s''}, d'' \cdot \mathrm{id}_{E_1 \times \cdots \times E_{g'}}) : \mathcal{A}_{s''} \to A_0$, where $d' = \left\lceil \sqrt{\deg \tilde{\psi}_{s'}} \right\rceil$ and $d'' = \left\lceil \sqrt{\deg \tilde{\psi}_{s''}} \right\rceil$. Recall that $\mathcal{A}_{s'}$, $\mathcal{A}_{s''}$, and A_0 are embedded into $\mathbb{P}^R$, inducing line bundles $\mathcal{L}_{s'}$ on $\mathcal{A}_{s'}$, $\mathcal{L}_{s''}$ on $\mathcal{A}_{s''}$, and L_0 on A_0. Let $\chi : \mathcal{A}_{s'} \times \mathcal{A}_{s''} \to A_0$ be defined by $\chi(y', y'') = b'\tilde{\phi}_{s'}(y') + b''\tilde{\phi}_{s''}(y'')$ and let $\mathrm{pr}_1 : \mathcal{A}_{s'} \times \mathcal{A}_{s''} \to \mathcal{A}_{s'}$ and $\mathrm{pr}_2 : \mathcal{A}_{s'} \times \mathcal{A}_{s''} \to \mathcal{A}_{s''}$ be the canonical projections. Set $a' = 4(b'd')^2$ and $a'' = 4(b''d'')^2$. The following hold:*

(i) For each $y \in \mathcal{A}_{s'} \times \mathcal{A}_{s''}$, there exists an injection from $\chi^* L_0$ into $\mathrm{pr}_1^* \mathcal{L}_{s'}^{\otimes 4a'} \otimes \mathrm{pr}_2^* \mathcal{L}_{s''}^{\otimes 4a''}$ that induces an isomorphism in an open neighbourhood of y. This injection can be chosen from a set of cardinality at most $(R+1)^5$.

(ii) The line bundle $\chi^* L_0$ is generated by a set of $R+1$ sections – the pull-backs of the homogeneous coordinates on $A_0 \subset \mathbb{P}^R$ –, mapping under the chosen injection to bihomogeneous polynomials in the coordinates given by the embedding $\mathcal{A}_{s'} \times \mathcal{A}_{s''} \hookrightarrow \mathbb{P}^R \times \mathbb{P}^R$ of bidegree $(4a', 4a'')$.

(iii) Furthermore, there exists a constant $\tilde{c}$, depending only on A_0, the family $\mathcal{A}$, and their (quasi-)projective immersions, such that the set of all coefficients

of these $R+1$ polynomials has height at most $a'\delta' + a''\delta''$, where $\delta' \leq \tilde{c}d'^{18}\max\{h_{\overline{S}}(s'), 1\}$, $\delta'' \leq \tilde{c}d''^{18}\max\{h_{\overline{S}}(s''), 1\}$.

PROOF. Define $\sigma_{b',b''} : A_0 \times A_0 \to A_0$ by $\sigma_{b',b''}(y', y'') = b'y' + b''y''$. We can show as in the proof of Lemma 4.4.6 that the embeddings of $E_0, E_1, \ldots, E_{g'}$ into $\mathbb{P}^5$ are projectively normal. It follows by repeated application of the Künneth formula (Proposition 9.2.4 in [77]) that the embedding of A_0 into $\mathbb{P}^R$ is projectively normal as well. We can therefore apply Proposition 5.2 from [150].

It yields a set of $(R+1)^3$ morphisms of invertible sheaves from $\sigma_{b',b''}^* L_0$ into $\mathrm{pr}_1^* L_0^{\otimes 4b'^2} \otimes \mathrm{pr}_2^* L_0^{\otimes 4b''^2}$ such that for each $z \in A_0 \times A_0$ one of them is an injection that restricts to an isomorphism in an open neighbourhood of z, where by abuse of notation pr_1 and pr_2 denote also the canonical projections $A_0 \times A_0 \to A_0$. We also obtain a set of $R+1$ sections – the pull-backs of the homogeneous coordinates on $A_0 \subset \mathbb{P}^R$ – that generate $\sigma_{b',b''}^* L_0$ such that by taking the union of the images of these sections under all injections, we obtain a set of at most $(R+1)^4$ bihomogeneous polynomials in the coordinates on $A_0 \times A_0$ of bidegree $(4b'^2, 4b''^2)$. Given a choice of injection, the set of all coefficients of the resulting $R+1$ polynomials has height bounded by $\tilde{c}_{A_0}(b'^2 + b''^2)$, where $\tilde{c}_{A_0}$ depends only on A_0 and its embedding into $\mathbb{P}^R$. We have $\chi = \sigma_{b',b''} \circ (\tilde{\phi}_{s'}, \tilde{\phi}_{s''})$.

Let $j \in \{1, \ldots, g'\}$. It follows from the proof of Proposition 5.2 in [150] (applied to $E_j \subset \mathbb{P}^5$) that there exist 6 sextuples of homogeneous polynomials in the coordinates on E_j of degree $4d'^2$ such that for each point of E_j one of the sextuples describes the multiplication-by-d' morphism in a Zariski open neighbourhood of that point and the set of all coefficients of any sextuple has height bounded by $c'd'^2$ for some constant c' that depends only on the E_i ($i = 1, \ldots, g'$) and their embeddings into $\mathbb{P}^5$.

It follows from Lemma 4.4.6 that there exist 6 sextuples of homogeneous polynomials in the coordinates on $\mathcal{E}_{s'}$ of degree $4d'^2$ such that for each point of $\mathcal{E}_{s'}$ one of the sextuples describes the isogeny $\tilde{\psi}_{s'}$ in a Zariski open neighbourhood of that point and the set of all coefficients of any sextuple has height bounded by $\tilde{c}(\deg \tilde{\psi}_{s'})^{10} \max\{h_{\overline{S}}(s'), 1\}$.

By choosing on each elliptic factor one of these sextuples and multiplying together all possible combinations of their entries, we find a collection of $6^g = R+1$ $(R+1)$-tuples of homogeneous polynomials in the coordinates on $\mathcal{A}_{s'}$ of degree $4d'^2$ such that for each point of $\mathcal{A}_{s'}$ one of these $(R+1)$-tuples describes the isogeny $\tilde{\phi}_{s'}$ in a Zariski open neighbourhood of that point with respect to the given embeddings into $\mathbb{P}^R$.

Because of the shape of the Segre embedding, the height of the family of coefficients of each $(R+1)$-tuple of polynomials can be bounded from above by $g'c'd'^2 + (g-g')\tilde{c}(\deg \tilde{\psi}_{s'})^{10} \max\{h_{\overline{S}}(s'), 1\}$ and thus by $\tilde{c}d'^{20} \max\{h_{\overline{S}}(s'), 1\}$ (after increasing $\tilde{c}$). We can do the same thing for $\tilde{\phi}_{s''}$ with d'' instead of d' and s'' instead of s'.

Now plugging in each of the $(R+1)^2$ combinations of these $(R+1)$-tuples of polynomials into the set of $(R+1)^4$ polynomials from the beginning of the proof gives us a set of $(R+1)^6$ bihomogeneous polynomials in the coordinates on $\mathcal{A}_{s'} \times \mathcal{A}_{s''}$ of bidegree $(4a', 4a'')$, which are the images of a set of $R+1$ sections that generate $\chi^* L_0$ under $(R+1)^5$ different morphisms of invertible sheaves from $\chi^* L_0$ to $\mathrm{pr}_1^* \mathcal{L}_{s'}^{\otimes 4a'} \otimes \mathrm{pr}_2^* \mathcal{L}_{s''}^{\otimes 4a''}$. Each possibility for the morphism corresponds to one of the $(R+1)^2$ combinations from above together with one of the $(R+1)^3$ possibilities from the beginning of the proof. For each $y \in \mathcal{A}_{s'} \times \mathcal{A}_{s''}$, one of these morphisms is an injection that restricts to an isomorphism in a neighbourhood of y. The sections are the pull-backs of the homogeneous coordinates on $A_0 \subset \mathbb{P}^R$.

We can bound the height of the family of coefficients of all $R+1$ polynomials corresponding to a choice of injection from above by

$$\begin{aligned}\tilde{c}_{A_0}(b'^2 + b''^2) + \tilde{c}(4b'^2 d'^{20} \max\{h_{\overline{S}}(s'), 1\} + 4b''^2 d''^{20} \max\{h_{\overline{S}}(s''), 1\}) \\ +4b'^2 \log \binom{R+4d'^2}{R} + 4b''^2 \log \binom{R+4d''^2}{R} \\ + \log \binom{R+4b'^2}{R} + \log \binom{R+4b''^2}{R}.\end{aligned}$$

Here the last four summands are a very crude upper bound for the logarithm of the number of monomials that one obtains after multiplying out and before combining like terms and hence also an upper bound for the logarithm of the maximal number of equal monomials that are obtained in this way. The lemma now follows (after increasing $\tilde{c}$ again) from estimating

$$\binom{R+4d'^2}{R} \leq (R+4d'^2)^R$$

and analogously for d'', b', and b''. $\square$

LEMMA 4.4.8. *Let $s_1, \ldots, s_m \in S$ and let $b_1, \ldots, b_m \in \mathbb{N}$. Let a_i, d_i, and $\tilde{\phi}_i$ ($i = 1, \ldots, m$) as well as $a = (a_1, \ldots, a_m)$ and $\mathcal{N}_a$ be defined as above. Let $\tilde{\Psi} : X_1 \times \cdots \times X_m \to A_0^{m-1}$ be the morphism given by $\tilde{\Psi}(y_1, \ldots, y_m) =$*

$$(b_1\tilde{\phi}_1(y_1) + b_2\tilde{\phi}_2(y_2), \ldots, b_{m-1}\tilde{\phi}_{m-1}(y_{m-1}) + b_m\tilde{\phi}_m(y_m)).$$

Recall that $X_i \subset \mathcal{A}_{s_i}$ ($i = 1, \ldots, m$) and A_0 are all embedded into $\mathbb{P}^R$. Let $q_1, \ldots, q_{m-1} : A_0^{m-1} \to A_0$ be the canonical projections. The following hold:

(i) For each $z \in X_1 \times \cdots \times X_m$, there exists an injection $\tilde{\Psi}^*(q_1^* L_0 \otimes \cdots \otimes q_{m-1}^* L_0) \hookrightarrow \mathcal{N}_a^{\otimes 8}$ that induces an isomorphism on an open neighbourhood of z. It can be chosen from a set of cardinality at most $(R+1)^{5m-3}$.

(ii) The line bundle $\tilde{\Psi}^*(q_1^* L_0 \otimes \cdots \otimes q_{m-1}^* L_0)$ is generated by $M = (R+1)^{m-1}$ sections – the pull-backs of the homogeneous coordinates on $A_0^{m-1} \subset (\mathbb{P}^R)^{m-1} \hookrightarrow \mathbb{P}^{M-1}$ –, mapping under the chosen injection to multihomogeneous polynomials of multidegree $8a = (8a_1, \ldots, 8a_m)$ in the multiprojective coordinates on $X_1 \times \cdots \times X_m \subset (\mathbb{P}^R)^m$.

(iii) Furthermore, there exists a constant $\tilde{c}$, depending only on A_0, the family $\mathcal{A}$, and their (quasi-)projective immersions, such that the set of all coefficients of these polynomials has height at most $\sum_{i=1}^m a_i \delta_i$, where $\delta_i \leq \tilde{c} d_i^{18} \max\{h_{\overline{S}}(s_i), 1\}$ ($i = 1, \ldots, m$).

PROOF. We apply Lemma 4.4.7 $m-1$ times with $(s', s'') = (s_i, s_{i+1})$ ($i = 1, \ldots, m-1$). We obtain $m-1$ systems of at most $(R+1)^6$ bihomogeneous polynomials each. Multiplying together all possible combinations of one section from each system gives us at most $(R+1)^{6(m-1)}$ multihomogeneous polynomials of multidegree $(4a_1, 8a_2, \ldots, 8a_{m-1}, 4a_m)$, corresponding to the union of the images of a system of $M = (R+1)^{m-1}$ sections that generates $\tilde{\Psi}^*(q_1^* L_0 \otimes \cdots \otimes q_{m-1}^* L_0)$ under each possible combination of the injections furnished by Lemma 4.4.7 (some of them might not remain injections, when pulled back to $X_1 \times \cdots \times X_m$, and we discard those). We multiply each polynomial by each combination of a $(4a_1)$-th power of one of the $R+1$ coordinates on X_1 and a $(4a_m)$-th power of one of the $R+1$ coordinates on X_m. This yields at most $(R+1)^{6m-4}$ multihomogeneous polynomials of multidegree $8a$, still corresponding to the union of the images of a system of M sections that generate $\tilde{\Psi}^*(q_1^* L_0 \otimes \cdots \otimes q_{m-1}^* L_0)$ under one of the thus obtained injections $\tilde{\Psi}^*(q_1^* L_0 \otimes \cdots \otimes q_{m-1}^* L_0) \hookrightarrow \mathcal{N}_a^{\otimes 8}$. The sections are the pull-backs of the homogeneous coordinates on $A_0^{m-1} \subset (\mathbb{P}^R)^{m-1} \hookrightarrow \mathbb{P}^{M-1}$.

By Lemma 4.4.7, we can bound the height of the family of coefficients of all multihomogeneous polynomials corresponding to one choice of injection by

$$\sum_{i=1}^{m} 2\tilde{c} a_i d_i^{18} \max\{h_{\overline{S}}(s_i), 1\} + \sum_{i=1}^{m} 2 \log \binom{R+4a_i}{R},$$

where the second summand is again a crude upper bound for the logarithm of the number of monomials obtained after multiplying out and before combining like terms and hence also an upper bound for the logarithm of the maximal number of equal monomials that are obtained in this way. The lemma follows from

$$\binom{R+4a_i}{R} \leq (R+4a_i)^R \quad (i = 1, \ldots, m)$$

after increasing $\tilde{c}$ again. □

Proof. (of Theorem 4.4.2) Putting together everything we did so far in this section, one can see that we have now proven Theorem 4.4.2. We summarize the proof: We divide $\Gamma \otimes \mathbb{R}$ into finitely many cones such that for each choice of ζ_i, ζ_{i+1} in one of these cones the inequality (4.4.14) is satisfied. This is possible since $\Gamma \otimes \mathbb{R}$ is a finite-dimensional vector space with respect to $\|\cdot\|$ and c_1 satisfies $c_1 \preceq 1$ by our choice of parameters and (4.4.4). If we prove a height bound of the desired form for each cone, we also get a global one by taking the maximum over the (finitely many) implicit constants.

If a cone contains only finitely many of the points p whose height we would like to bound, the desired bound holds trivially. If a cone contains infinitely many such points, we can suppose that we can find among them points $x_1, \ldots, x_m$ which satisfy (4.4.8), (4.4.9), and (4.4.10); otherwise, a bound of the desired form follows from (4.4.1), (4.4.5), Lemma 4.4.1, Lemma 4.4.3, and the choice of parameters in the generalized Vojta-Rémond inequality. But then, by (4.4.12), (4.4.13), Lemma 4.4.5, and Lemma 4.4.8, the generalized Vojta-Rémond inequality in the form of Theorem A.1.1 can be applied to yield a contradiction with (4.4.16). So Theorem 4.4.2 follows. □

4.5. Application of the Pila-Zannier strategy

In this section, we will apply the Pila-Zannier strategy to deduce the following Proposition 4.5.1 from Theorem 4.4.2. This part is very similar to the case of curves that was treated in Chapter 3 and no substantial new difficulties appear, so we often refer to that chapter and try to keep the exposition short. In particular, we will use terminology from the theory of o-minimal structures without further explanation and refer the reader to Section 3.5 for definitions. In order to speak of definable subsets of powers of $\mathbb{C}$, we identify $\mathbb{C}$ with $\mathbb{R}^2$ through use of the maps Re and Im.

PROPOSITION 4.5.1. *Let $\mathcal{A} \to S$ and A_0 be as in Section 4.4. Let $\mathcal{V} \subset \mathcal{A}$ be an irreducible subvariety that dominates S. Let $\mathcal{U}$ be the set of points $p \in \mathcal{V} \cap \mathcal{A}_\Gamma$ that do not lie in a translate of a positive-dimensional abelian subvariety of $\mathcal{A}_{\pi(p)}$ contained in $\mathcal{V}_{\pi(p)}$. Then there exists a subgroup $\Gamma' \subset \mathcal{A}_\xi^{\overline{\bar{\mathbb{Q}}(S)}/\bar{\mathbb{Q}}}(\bar{\mathbb{Q}})$ of finite rank such that for all but finitely many $p \in \mathcal{U}$ there exists an irreducible curve $\mathcal{C}$ such that $p \in \mathcal{C}$, $\pi(\mathcal{C}) = S$, $\mathcal{C} \subset \mathcal{V}$, and $\mathcal{C}_\xi \subset (\mathcal{A}_\xi)_{\mathrm{tors}} + \mathrm{Tr}\,(\Gamma')$.*

PROOF. As in Section 4.4, we can assume without loss of generality that Γ is invariant under $\mathrm{End}(A_0)$. For each $s \in S$ such that A_0 and $\mathcal{A}_s$ are isogenous, we fix an isogeny $\phi_s : A_0 \to \mathcal{A}_s$ as in Section 4.4. Let p be a point in $\mathcal{U}$. By Lemma 3.2.2, we have $p = \phi_{\pi(p)}(\gamma)$ for some $\gamma \in \Gamma$. By Theorem 4.4.2, we have $h_{\pi(p)}(p) \preceq [K(p) : K]$, where K is as in Section 4.4. It follows from Lemma 4.4.1, Lemma 4.4.3, (4.4.1), and (4.4.5) that also $\widehat{h}_{A_0}(\gamma) \preceq [K(p) : K]$. If N is the smallest natural number such that $N\gamma = a_1\gamma_1 + \cdots + a_r\gamma_r \in \bigoplus_{i=1}^r \mathbb{Z}\gamma_i$, then we can show as in Proposition 3.4.3 that $\max\{N, |a_1|, \ldots, |a_r|\} \preceq [K(p) : K]$. If $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/K)$, then we deduce that $\sigma(p) = \sigma\left(\phi_{\pi(p)}\right)(\sigma(\gamma))$, where σ acts on algebraic points and maps in the usual way and $\sigma\left(\phi_{\pi(p)}\right)$ is an isogeny between A_0 and $\mathcal{A}_{\pi(\sigma(p))}$. It follows that there is $\gamma_\sigma \in \Gamma$ with $\sigma(p) = \phi_{\pi(\sigma(p))}(\gamma_\sigma)$. Since $h_{\pi(p)}(p) = h_{\pi(\sigma(p))}(\sigma(p))$ and $[K(p) : K] = [K(\sigma(p)) : K]$, we find that also

$$\max\{N_\sigma, |a_{\sigma,1}|, \ldots, |a_{\sigma,r}|\} \preceq [K(p) : K] \tag{4.5.1}$$

for the analogous quantities for γ_σ.

There is a uniformization map $e : \mathbb{H} \times \mathbb{C} \to \mathcal{E}(\mathbb{C})$. Its restriction to

$$\{(\tau, z) \in \mathcal{F} \times \mathbb{C}; z = x + y\tau, x, y \in [0, 1)\}$$

is surjective and definable in the o-minimal structure $\mathbb{R}_{\mathrm{an,exp}}$, where $\mathcal{F}$ is a fundamental domain in $\mathbb{H}$ for the action of a certain subgroup of $\mathrm{SL}_2(\mathbb{Z})$ of finite index. For details, see for example Sections 2 and 3 of [132]; the definability follows from [127].

We choose $\tau_0, \tau_1, \ldots, \tau_{g'} \in \mathcal{F}$ such that $E_i(\mathbb{C})$ is isomorphic to $\mathbb{C}/(\mathbb{Z}+\tau_i\mathbb{Z})$. There are uniformization maps $e_i : \mathbb{C} \to E_i(\mathbb{C})$ with kernel $\mathbb{Z}+\tau_i\mathbb{Z}$ that are definable in the o-minimal structure $\mathbb{R}_{\mathrm{an}}$ (and hence in $\mathbb{R}_{\mathrm{an,exp}}$) when restricted to $\{x + y\tau_i; x, y \in [0, 1)\}$ $(i = 0, \ldots, g')$.

We can define a uniformization map $\exp : \mathbb{H} \times \mathbb{C}^g \to \mathcal{A}(\mathbb{C}) = (\mathcal{E} \times_{Y(2)} \cdots \times_{Y(2)} \mathcal{E})(\mathbb{C}) \times E_1(\mathbb{C}) \times \cdots \times E_{g'}(\mathbb{C})$ by

$$\exp(\tau, z_1, \ldots, z_g) = (e(\tau, z_1), \ldots, e(\tau, z_{g-g'}), e_1(z_{g-g'+1}), \ldots, e_{g'}(z_g)).$$

It has a fundamental domain

$$\begin{aligned} U = \{(\tau, z_1, \ldots, z_g) \in \mathcal{F} \times \mathbb{C}^g; z_i = x_i + y_i\tau, z_{g-g'+j} = x_{g-g'+j} + y_{g-g'+j}\tau_j, \\ i = 1, \ldots, g - g', j = 1, \ldots, g', x_1, \ldots, x_g, y_1, \ldots, y_g \in [0, 1)\}, \end{aligned}$$

restricted to which it is surjective and definable in $\mathbb{R}_{\mathrm{an,exp}}$. In order to speak of definability, we have to fix an embedding of $\mathcal{A}(\mathbb{C})$ into projective space and we can take the one from Section 4.4.

We set

$$P = \begin{pmatrix} \tau_1 & & \\ & \ddots & \\ & & \tau_{g'} \end{pmatrix}.$$

For $\tau \in \mathbb{H}$, we set

$$\Pi_\tau = \begin{pmatrix} \tau I_{g-g'} & \\ & P \end{pmatrix}$$

and

$$\Omega_\tau = \begin{pmatrix} \Pi_\tau & I_g \end{pmatrix}.$$

We can find $(\tau_\sigma, x_\sigma) \in \mathbb{H} \times [0,1)^{2g}$ such that $(\tau_\sigma, \Omega_{\tau_\sigma} x_\sigma) \in U$ and $\exp(\tau_\sigma, \Omega_{\tau_\sigma} x_\sigma) = \sigma(p)$.

We can also define a uniformization map $\exp_0 : \mathbb{C}^g \to A_0(\mathbb{C}) = E_0(\mathbb{C})^{g-g'} \times E_1(\mathbb{C}) \times \cdots \times E_{g'}(\mathbb{C})$ by

$$\exp_0(z_1, \ldots, z_g) = (e_0(z_1), \ldots, e_0(z_{g-g'}), e_1(z_{g-g'+1}), \ldots, e_{g'}(z_g)),$$

where we consider $A_0(\mathbb{C})$ as embedded into projective space as in Section 4.4. The uniformization map is then surjective and definable in $\mathbb{R}_{\mathrm{an}}$ when restricted to a fundamental domain

$$\{(x_1 + y_1\tau_0, \ldots, x_{g-g'} + y_{g-g'}\tau_0, x_{g-g'+1} + y_{g-g'+1}\tau_1, \ldots, x_g + y_g\tau_{g'}); \\ x_1, \ldots, x_g, y_1, \ldots, y_g \in [0,1)\}.$$

The points $\gamma_1, \ldots, \gamma_r$ have pre-images $\tilde{\gamma}_1, \ldots, \tilde{\gamma}_r$ in this fundamental domain under $\exp_0$.

The isogeny $\phi_{\pi(\sigma(p))}$ lifts under the uniformizations $\exp_0$ and $\exp(\tau_\sigma, \cdot)$ to a linear map from $\mathbb{C}^g$ to $\mathbb{C}^g$ given by

$$\tilde{\alpha}_\sigma = \begin{pmatrix} \alpha_\sigma I_{g-g'} & \\ & d_{\pi(\sigma(p))} I_{g'} \end{pmatrix},$$

where $\phi_{\pi(\sigma(p))} = (\psi_{\pi(\sigma(p))}, \ldots, \psi_{\pi(\sigma(p))}, d_{\pi(\sigma(p))} \cdot \mathrm{id}_{E_1 \times \cdots \times E_{g'}})$ and $\alpha_\sigma \in \mathbb{C}$ is the analytic representation of $\psi_{\pi(\sigma(p))}$ with respect to the given uniformizations of $E_0(\mathbb{C})$ and $\mathcal{E}_{\pi(\sigma(p))}(\mathbb{C})$. There exists a matrix $\Psi_\sigma = \begin{pmatrix} b_{\sigma,1} b_{\sigma,2} \\ b_{\sigma,3} b_{\sigma,4} \end{pmatrix} \in \mathrm{M}_2(\mathbb{Z}) \cap \mathrm{GL}_2(\mathbb{Q})$ of determinant $\deg \psi_{\pi(\sigma(p))}$ such that

$$\alpha_\sigma \begin{pmatrix} \tau_0 & 1 \end{pmatrix} = \begin{pmatrix} \tau_\sigma & 1 \end{pmatrix} \Psi_\sigma. \tag{4.5.2}$$

It follows from (4.4.1) and the definition of $d_{\pi(\sigma(p))}$ that

$$d_{\pi(\sigma(p))} \leq \sqrt{\deg \psi_{\pi(\sigma(p))}} \preceq [K(p) : K]. \tag{4.5.3}$$

Since $\exp(\tau_\sigma, \Omega_{\tau_\sigma} x_\sigma) = \sigma(p) = \phi_{\pi(\sigma(p))}(\gamma_\sigma)$, we deduce that $\exp_0(\tilde{\alpha}_\sigma^{-1} \Omega_{\tau_\sigma} x_\sigma) \in \gamma_\sigma + \ker \phi_{\pi(\sigma(p))}$. As $\ker \phi_{\pi(\sigma(p))}$ is annihilated by $(\det \Psi_\sigma)\, d_{\pi(\sigma(p))}$, we deduce that $\exp_0((\det \Psi_\sigma)\, d_{\pi(\sigma(p))} \tilde{\alpha}_\sigma^{-1} \Omega_{\tau_\sigma} x_\sigma) = (\det \Psi_\sigma)\, d_{\pi(\sigma(p))} \gamma_\sigma$. It follows that

$$(\det \Psi_\sigma)\, d_{\pi(\sigma(p))} \left(N_\sigma \tilde{\alpha}_\sigma^{-1} \Omega_{\tau_\sigma} x_\sigma - a_{\sigma,1} \tilde{\gamma}_1 - \cdots - a_{\sigma,r} \tilde{\gamma}_r \right) \in \ker \exp_0,$$

so there exists $R_\sigma \in \mathbb{Z}^{2g}$ such that

$$(\det \Psi_\sigma)\, d_{\pi(\sigma(p))} \left(N_\sigma \tilde{\alpha}_\sigma^{-1} \Omega_{\tau_\sigma} x_\sigma - a_{\sigma,1}\tilde{\gamma}_1 - \cdots - a_{\sigma,r}\tilde{\gamma}_r\right) = \Omega_{\tau_0} R_\sigma. \tag{4.5.4}$$

It follows from (4.5.2) that

$$\tau_0 = \frac{\tau_\sigma b_{\sigma,1} + b_{\sigma,3}}{\tau_\sigma b_{\sigma,2} + b_{\sigma,4}}$$

and therefore

$$\tau_\sigma = \frac{\tau_0 b_{\sigma,4} - b_{\sigma,3}}{-\tau_0 b_{\sigma,2} + b_{\sigma,1}}. \tag{4.5.5}$$

Since $\tau_\sigma \in \mathcal{F}$, Theorem 1.1 in [125] with $\mathbf{G} = \mathrm{GL}_2$, $n = 2$, and $\rho = \mathrm{id}_{\mathrm{GL}_2}$ shows that $\|\Psi_\sigma\| \preceq \det \Psi_\sigma = \deg \psi_{\pi(\sigma(p))}$ and hence

$$\|\Psi_\sigma\| \preceq [K(p) : K] \tag{4.5.6}$$

by (4.4.1).

For $D \in \mathbb{N}$ and $B = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix} \in \mathrm{M}_2(\mathbb{Z})$, we define

$$\beta_i(B) = \begin{pmatrix} B_i I_{g-g'} & 0 \\ 0 & 0 \end{pmatrix} \in \mathrm{M}_g(\mathbb{Z}) \quad (i = 1, \ldots, 4),$$

$$\beta_5(D) = \begin{pmatrix} 0 & 0 \\ 0 & D I_{g'} \end{pmatrix} \in \mathrm{M}_g(\mathbb{Z})$$

and

$$\beta(B, D) = \begin{pmatrix} \beta_1(B) + \beta_5(D) & \beta_2(B) \\ \beta_3(B) & \beta_4(B) + \beta_5(D) \end{pmatrix} \in \mathrm{M}_{2g}(\mathbb{Z}).$$

We can deduce from (4.5.2) that

$$\tilde{\alpha}_\sigma \Omega_{\tau_0} = \Omega_{\tau_\sigma} \beta\left(\Psi_\sigma, d_{\pi(\sigma(p))}\right).$$

Here, $\beta\left(\Psi_\sigma, d_{\pi(\sigma(p))}\right)$ is invertible and it follows that

$$\tilde{\alpha}_\sigma^{-1} \Omega_{\tau_\sigma} = \Omega_{\tau_0} \beta\left(\Psi_\sigma, d_{\pi(\sigma(p))}\right)^{-1}. \tag{4.5.7}$$

Together with (4.5.4), this implies that

$$(\det \Psi_\sigma)\, d_{\pi(\sigma(p))} \left(N_\sigma \Omega_{\tau_0} \beta\left(\Psi_\sigma, d_{\pi(\sigma(p))}\right)^{-1} x_\sigma - a_{\sigma,1}\tilde{\gamma}_1 - \cdots - a_{\sigma,r}\tilde{\gamma}_r\right) = \Omega_{\tau_0} R_\sigma. \tag{4.5.8}$$

It now follows from (4.5.1), (4.5.3), and (4.5.6) as well as $x_\sigma \in [0,1)^{2g}$ that

$$\|R_\sigma\| \preceq [K(p) : K] \tag{4.5.9}$$

since we can solve (4.5.8) for R_σ by conjugating and obtaining $\begin{pmatrix} \Omega_{\tau_0} \\ \overline{\Omega_{\tau_0}} \end{pmatrix} R_\sigma$ on the right-hand side, where $\begin{pmatrix} \Omega_{\tau_0} \\ \overline{\Omega_{\tau_0}} \end{pmatrix}$ is invertible.

We set $X = \exp|_U^{-1}(\mathcal{V}(\mathbb{C}))$. From now on, "definable" will always mean "definable in the o-minimal structure $\mathbb{R}_{\text{an,exp}}$". Consider the definable set

$$Z = \{(A_1, \ldots, A_r, M, R, B_1, B_2, B_3, B_4, D, \tau, x) \in \mathbb{R}^{r+1+2g+5} \times \mathbb{H} \times \mathbb{R}^{2g};$$
$$(\tau, \Omega_\tau x) \in X, B = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix}, (\det B)D \neq 0, \tau(-B_2\tau_0 + B_1) = B_4\tau_0 - B_3,$$
$$M > 0, \Omega_{\tau_0} R = (\det B)D\left(M\Omega_{\tau_0}\beta(B, D)^{-1}x - A_1\tilde{\gamma}_1 - \cdots - A_r\tilde{\gamma}_r\right)\}.$$

Let $\pi_1 : Z \to \mathbb{R}^{r+1+2g+5}$ and $\pi_2 : Z \to \mathbb{H} \times \mathbb{R}^{2g}$ be the canonical projections and let Σ be the set of points

$$(a_{\sigma,1}, \ldots, a_{\sigma,r}, N_\sigma, R_\sigma, b_{\sigma,1}, \ldots, b_{\sigma,4}, d_{\pi(\sigma(p))}, \tau_\sigma, x_\sigma) \quad (\sigma \in \operatorname{Gal}(\bar{\mathbb{Q}}/K)).$$

It follows from (4.5.5) and (4.5.8) that $\Sigma \subset Z$. Furthermore, we have $|\pi_2(\Sigma)| = [K(p) : K]$ and it follows from (4.5.1), (4.5.3), (4.5.6), and (4.5.9) that

$$\max\{|a_{\sigma,1}|, \ldots, |a_{\sigma,r}|, N_\sigma, \|R_\sigma\|, |b_{\sigma,1}|, \ldots, |b_{\sigma,4}|, d_{\pi(\sigma(p))}\} \preceq [K(p) : K].$$

If $[K(p) : K]$ is sufficiently big (which by Lemma 4.4.1, Theorem 4.4.2, and Northcott's theorem excludes only finitely many $p \in \mathcal{U}$), we can apply Corollary 7.2 from [70] and find that there exists a continuous function $\alpha : [0, 1] \to Z$ such that $\pi_1 \circ \alpha$ is semialgebraic, $\pi_2 \circ \alpha$ is non-constant, $\pi_2(\alpha(0)) \in \pi_2(\Sigma)$, and $\alpha|_{(0,1)}$ is real analytic; note that $\mathbb{R}_{\text{an,exp}}$ admits analytic cell decomposition by [181]. In the following, we use the variables $\tau, B, \ldots$ to denote the corresponding coordinate functions on Z. It follows from $\tau = \frac{B_4\tau_0 - B_3}{-B_2\tau_0 + B_1}$ and

$$\Omega_{\tau_0}\beta(B, D)^{-1}x = M^{-1}\left(A_1\tilde{\gamma}_1 + \cdots + A_r\tilde{\gamma}_r + (\det B)^{-1}D^{-1}\Omega_{\tau_0}R\right)$$

that α itself is a semialgebraic map. Note that we can solve the last equation for x by conjugating and obtaining the invertible matrix $\begin{pmatrix} \Omega_{\tau_0} \\ \overline{\Omega_{\tau_0}} \end{pmatrix}$. Let $\psi : \mathbb{H} \times \mathbb{R}^{2g} \to \mathbb{H} \times \mathbb{C}^g$ be defined by $\psi(\tau, x) = (\tau, \Omega_\tau x)$. It follows that $\delta = \psi \circ \pi_2 \circ \alpha : [0, 1] \to X$ is a non-constant continuous semialgebraic map and $\exp(\delta(0))$ is equal to some Galois conjugate of p.

We can now deduce from Theorem 5.1 in [133] and Lemma 4.1 in [134] that some Galois conjugate of p and hence also p itself lies in a positive-dimensional weakly special subvariety of the product of elliptic modular surfaces and elliptic curves $\mathcal{E}(\mathbb{C})^{g-g'} \times E_1(\mathbb{C}) \times \cdots \times E_{g'}(\mathbb{C})$ that is contained in $\mathcal{V}(\mathbb{C})$. In order to apply Theorem 5.1 from [133], we view the factors $E_i(\mathbb{C})$ $(i = 1, \ldots, g')$ as fibers of some elliptic modular surfaces as defined in Section 2.2 of [133]. For the definition of a weakly special subvariety, we refer to Section 4 of [133]. The analogue of Theorem 5.1 in [133] for arbitrary connected mixed Shimura varieties has been proven later by Gao in [52].

At the same time, the point p cannot lie in a translate of a positive-dimensional abelian subvariety of $\mathcal{A}_{\pi(p)}(\mathbb{C})$ contained in $\mathcal{V}_{\pi(p)}(\mathbb{C})$. It follows that p must belong to an irreducible curve $\mathcal{C}$, a priori defined over $\mathbb{C}$, such that $\emptyset \neq \mathcal{C}_\xi \subset (\mathcal{A}_\xi)_{\text{tors}} + \operatorname{Tr}\left(\mathcal{A}_\xi^{\overline{\bar{\mathbb{Q}}(S)}/\bar{\mathbb{Q}}}(\mathbb{C})\right)$ (we base change freely between $\bar{\mathbb{Q}}$ and $\mathbb{C}$ and between $\overline{\bar{\mathbb{Q}}(S)}$ and an algebraic closure of the function field of the base change of S to $\mathbb{C}$). As $p \in \mathcal{A}_\Gamma$, we can first replace $\mathcal{A}_\xi^{\overline{\bar{\mathbb{Q}}(S)}/\bar{\mathbb{Q}}}(\mathbb{C})$ by $\mathcal{A}_\xi^{\overline{\bar{\mathbb{Q}}(S)}/\bar{\mathbb{Q}}}(\bar{\mathbb{Q}})$ – in particular, $\mathcal{C}$ is defined over $\bar{\mathbb{Q}}$ – and then $\mathcal{A}_\xi^{\overline{\bar{\mathbb{Q}}(S)}/\bar{\mathbb{Q}}}(\bar{\mathbb{Q}})$ by a subgroup Γ' of finite rank. □

4.6. Proof of Theorem 4.1.2

We can assume without loss of generality that $A_0 = E_0^{g-g'} \times E_1 \times \cdots \times E_{g'}$ for elliptic curves as in the hypothesis of Theorem 4.1.2. We can also assume that $\pi(\mathcal{V}) = S$ since otherwise $\mathcal{V}$ is contained in $\mathcal{A}_s$ for some $s \in S$ and Theorem 4.1.2 then becomes an instance of the Mordell-Lang conjecture, proven by Vojta, Faltings, and Hindry.

It then follows that infinitely many fibers of $\mathcal{A}$ are pairwise isogenous. We deduce from Theorem 3 in [69] by Habegger-Pila that the generic fiber of $\mathcal{A}$ must be isogenous (over $\overline{\bar{\mathbb{Q}}(S)}$) to $E^{g-g''} \times E_1' \times \cdots \times E_{g''}'$, where E is an elliptic curve defined over $\overline{\bar{\mathbb{Q}}(S)}$ and $E_1', \ldots, E_{g''}'$ are elliptic curves defined over $\bar{\mathbb{Q}}$. The fact that infinitely many fibers of $\mathcal{A}$ are isogenous to A_0 and that $\operatorname{Hom}\left(\mathcal{A}_\xi^{\overline{\bar{\mathbb{Q}}(S)}/\bar{\mathbb{Q}}}, E_0\right) = \{0\}$ and therefore $\operatorname{Hom}(E_1' \times \cdots \times E_{g''}', E_0) = \{0\}$ as well as $\operatorname{Hom}(E_i, E_0) = \{0\}$ $(i = 1, \ldots, g')$ implies that $g' = g''$. In particular, we have $g' < g$ since $\mathcal{A}$ is not isotrivial. We can deduce furthermore that after a permutation of $E_1, \ldots, E_{g'}$ E_i' is isogenous to E_i $(i = 1, \ldots, g')$. We can then assume without loss of generality that $E_i' = E_i$ $(i = 1, \ldots, g')$.

Consider the set of isomorphism classes of pairs $(\mathcal{A} \to S, A_0)$ such that

1. S is a smooth and irreducible curve,
2. $\mathcal{A}$ is not isotrivial,
3. $\mathcal{A}_\xi$ is isogenous (over $\overline{\bar{\mathbb{Q}}(S)}$) to $E^{g-g'} \times E_1 \times \cdots \times E_{g'}$ for $g' \geq 0$, an elliptic curve E defined over $\overline{\bar{\mathbb{Q}}(S)}$, and elliptic curves $E_1, \ldots, E_{g'}$ defined over $\bar{\mathbb{Q}}$ that satisfy $\operatorname{Hom}(E_i, E_j) = \{0\}$ $(i \neq j)$, and
4. A_0 is isogenous to $E_0^{g-g'} \times E_1 \times \cdots \times E_{g'}$, where E_0 is an elliptic curve defined over $\bar{\mathbb{Q}}$, $\operatorname{Hom}(E_0, E_i) = \{0\}$ $(i = 1, \ldots, g')$, and either $g - g' = 1$ or $\operatorname{End}(E_0) = \mathbb{Z}$.

This set is stable in the sense of Definition 4.3.1. By Proposition 4.3.2, we can therefore assume that the union of all translates of positive-dimensional abelian subvarieties of $\mathcal{A}_s$ that are contained in $\mathcal{V}_s$ for some $s \in S$ is not Zariski dense in $\mathcal{V}$.

After replacing S by an open subset of a finite cover, $\mathcal{A}$ by its corresponding pull-back, and $\mathcal{V}$ by an irreducible component of its pull-back, we can assume that there exists an elliptic scheme $\mathcal{E}'$ over S and a surjective homomorphism $\alpha : \mathcal{A} \to \mathcal{E}' \times_S \cdots \times_S \mathcal{E}' \times_S (E_1 \times \cdots \times E_{g'} \times S)$ of abelian schemes over S that restricts to an isogeny on each fiber. For this, we use our hypothesis on the generic fiber of $\mathcal{A}$ together with Proposition 8 from Section 1.2 and Theorem 3 from Section 1.4 of [25].

Arguing along the lines of the proof of Lemma 5.4 in [68], we can then assume that $\mathcal{A} = (\mathcal{E} \times_S \cdots \times_S \mathcal{E}) \times_S (E_1 \times \cdots \times E_{g'} \times S)$, where $S = Y(2)$, $\mathcal{E}$ is the Legendre family, and $E_1, \ldots, E_{g'}$ are some fixed elliptic curves over $\bar{\mathbb{Q}}$. Thus, we are in the situation of Sections 4.4 and 4.5. Since the union of all translates of positive-dimensional abelian subvarieties of $\mathcal{A}_s$ that are contained in $\mathcal{V}_s$ for some

$s \in S$ is not Zariski dense in $\mathcal{V}$ and a finite set of points is not Zariski dense in $\mathcal{V}$, it follows from Proposition 4.5.1 that there exists a subgroup $\Gamma' \subset \mathcal{A}_{\xi}^{\overline{\bar{\mathbb{Q}}(S)}/\bar{\mathbb{Q}}}(\bar{\mathbb{Q}})$ of finite rank such that the irreducible curves $\mathcal{C}$ that are contained in $\mathcal{V}$ and satisfy $\emptyset \neq \mathcal{C}_{\xi} \subset (\mathcal{A}_{\xi})_{\mathrm{tors}} + \mathrm{Tr}\,(\Gamma')$ lie Zariski dense in $\mathcal{V}$. We can then apply Lemma 4.3.4 to finish the proof. □

CHAPTER 5

Metalegomena about unlikely intersections with isogeny orbits

Horseness is the whatness of allhorse.

J. Joyce, *Ulysses*

5.1. Sketch of the function field case

In this section, we indicate how our proof of Theorem 3.1.3 has to be modified if S, $\mathcal{A}$, $\mathcal{C}$, A_0, and Γ are no longer defined over $\bar{\mathbb{Q}}$, but over some field F of characteristic 0 that is finitely generated over $\mathbb{Q}$. Everything except for Section 3.4 can be left essentially unmodified. In Section 3.4, the following modifications are necessary: In general, the Weil height needs to be replaced by the Moriwaki height with respect to a big polarization of F as defined in [115]. Fundamental properties of the Moriwaki height machine, one of them due to Wazir [189], have been collected in Proposition A.1 in [55]. In particular, given a polarization of F, an abelian variety A over F, and a symmetric ample line bundle on A there exists an associated canonical height $h : A(\bar{F}) \to [0, \infty)$, where $\bar{F}$ is a fixed algebraic closure of F.

We can then prove the analogues of Lemmata 3.4.1 and 3.4.2 using Orr's Theorem 5.1 from [124] instead of the Masser-Wüstholz theorem and replacing Pazuki's theorem on the difference between the Faltings height and the Theta height by Proposition 4.1 in [116] as well as replacing the bound for the difference of the Faltings heights of isogenous abelian varieties in terms of the degree of the isogeny by Proposition 3.2 in [116].

Our explicit argument in the proof of Lemma 3.4.2 to obtain an upper bound for $\widehat{h}_s(p)$ in terms of $h_s(p)$ and $h_{\overline{S}}(s)$ also goes through by Proposition 3.3.7(1) in [115] and the shape of the naive Moriwaki height on projective space (see Section 3.2 of [115]). Masser's upper bound for the order of a torsion point in terms of its degree can be recovered by a specialization argument since specialization from characteristic 0 to characteristic 0 (given good reduction) is injective on the torsion subgroup. All of this has already been carried out in detail by Gao in [51].

We do not know if Masser's lower bound for the canonical height of a non-torsion point can be recovered in general and we do not know how to overcome this obstacle in proving Theorem 3.1.2. However, in order to prove Theorem 3.1.3, Proposition 3.4.3 is only needed in the special case $B_0 = \{0\}$. In this case, the lower bound is actually only ever needed for a non-torsion point that belongs to a fixed subgroup of finite rank. In this situation, the lower bound can be deduced from Kummer theory and a specialization argument as the following lemma shows. Its proof does not even need Masser's lower bound over number fields. For the proof of part (iii) in Proposition 3.4.3 only elementary diophantine approximation (and the above-mentioned bound for the order of a torsion point in terms of its degree) is then necessary, but the results of neither [70] nor [156] have to be used. This has also been carried out in detail by Gao in [51], albeit formulated slightly differently. We have chosen this formulation because of its conceptual similarity to our proof of Proposition 3.4.3 for B_0 arbitrary in the number field case.

LEMMA 5.1.1. *Let F be a field of characteristic 0 that is finitely generated over $\mathbb{Q}$. Let A be an abelian variety defined over F. Let h be a canonical Moriwaki height on $A(\bar{F})$ with respect to a big polarization of F and a symmetric ample line bundle on A. Let G be a subgroup of $A(\bar{F})$ of finite rank. There exist positive constants c and κ, depending only on A, F, G, and h, such that $h(p) \geq c[F(p) : F]^{-\kappa}$ for every non-torsion point $p \in G$.*

PROOF. We can assume without loss of generality that G is the division closure of $\mathbb{Z}g_1 + \cdots + \mathbb{Z}g_n$, where $g_1, \ldots, g_n \in A(\bar{F})$ are $\mathbb{Z}$-linearly independent. After replacing F by a finite extension, we can assume that $g_1, \ldots, g_n \in A(F)$. As h satisfies the Northcott property by Theorem 4.3 in [115] and $h(p) = 0$ only if p is a torsion point by Proposition 3.4.1(3) in [115], we know that $h(p) \geq c_1 > 0$ for all non-torsion points $p \in A(F)$, where c_1 depends only on A, h, and F.

Let now $p \in G$ be an arbitrary non-torsion point. There exists some natural number $N(p)$ such that $N(p)p \in \mathbb{Z}g_1 + \cdots + \mathbb{Z}g_n \subset A(F)$ and we take $N(p)$ minimal with $N(p)p \in \mathbb{Z}g_1 + \cdots + \mathbb{Z}g_n$. As the canonical height is quadratic (see Section 3.4 of [115]), it follows from $N(p)p \in A(F)$ that $h(p) = h(N(p)p)N(p)^{-2} \geq c_1 N(p)^{-2}$.

By "spreading out", we can view A as the generic fiber of an abelian scheme $\tilde{A} \to V$ defined over a number field $k \subset \bar{\mathbb{Q}}$ such that V is a geometrically irreducible k-variety with function field $k(V) = F$ (see Theorem 3.2.1 and Table 1 on pp. 306–307 in [144]). After maybe shrinking V to an open Zariski dense subset, we can assume that V is smooth and the g_i extend to sections $V \to \tilde{A}$. For every $g \in \mathbb{Z}g_1 + \cdots + \mathbb{Z}g_n$, we denote the section extending it by g as well.

By the Main Theorem and Scholium 1 in [96], there exists $q \in V(\bar{\mathbb{Q}})$ such that the specialization homomorphism $\mathbb{Z}g_1 + \cdots + \mathbb{Z}g_n \to \mathbb{Z}(g_1)_q + \cdots + \mathbb{Z}(g_n)_q$ is injective, where g_q denotes the value of $g \in \mathbb{Z}g_1 + \cdots + \mathbb{Z}g_n$ at q. After maybe enlarging k (and F), we can assume that $q \in V(k)$. We use a subscript q to denote fibers over q.

Choose a k-irreducible (but not necessarily geometrically irreducible) component $\mathcal{P}$ of $[N(p)]^{-1}((N(p)p)(V))$ such that the geometric generic fiber of $\mathcal{P}$ contains p. Here $[N] : \tilde{A} \to \tilde{A}$ denotes the multiplication-by-N morphism for $N \in \mathbb{Z}$. By [111], Proposition 20.7, the morphism $[N(p)]$ is finite and étale and therefore the morphism $[N(p)]^{-1}((N(p)p)(V)) \to V$ is finite and étale as well. Hence, the variety $[N(p)]^{-1}((N(p)p)(V))$ is smooth as V is smooth. It follows that the irreducible components of $[N(p)]^{-1}((N(p)p)(V))$ are pairwise disjoint. In particular, $\mathcal{P}$ is open and closed in $[N(p)]^{-1}((N(p)p)(V))$ and therefore the morphism $\mathcal{P} \to V$ is finite and étale as well. Therefore, it is finite locally free as defined in Section 12.6 of [58] and its degree is equal to $[F(p) : F]$ by construction. Hence the cardinality of its geometric fibers is bounded from above by $[F(p) : F]$ by [58], Proposition 12.21. We choose some $p_q \in \mathcal{P}_q(\bar{\mathbb{Q}})$. It follows that $[k(p_q) : k] \leq [F(p) : F]$.

We claim that $N(p) = N(p_q)$, where $N(p_q)$ is the smallest natural number such that $N(p_q)p_q \in \mathbb{Z}(g_1)_q + \cdots + \mathbb{Z}(g_n)_q$. By construction, we have that $N(p_q)$ divides $N(p)$. It follows from the injectivity of the specialization homomorphism that $N(p)p = N(p)N(p_q)^{-1}g$ for $g \in \mathbb{Z}g_1 + \cdots + \mathbb{Z}g_n$ with $g_q = N(p_q)p_q$. If $N(p_q) < N(p)$, this implies that $[N(p)]^{-1}((N(p)p)(V))$ has two distinct k-irreducible components that intersect each other in p_q (one of them is $\mathcal{P}$, the other is a k-irreducible component of $[N(p_q)]^{-1}(g(V))$). We obtain a contradiction with the smoothness of $[N(p)]^{-1}((N(p)p)(V))$.

Set $G'_q = \mathbb{Z}(g_1)_q + \cdots + \mathbb{Z}(g_n)_q + \mathbb{Z}p_q$. Now $N(p_q) \leq [G'_q \cap \tilde{A}_q(k) : \mathbb{Z}(g_1)_q + \cdots + \mathbb{Z}(g_n)_q]\tilde{N}(p_q)$, where $\tilde{N}(p_q)$ is the smallest natural number such that $\tilde{N}(p_q)p_q \in \tilde{A}_q(k)$. The first factor here is bounded by the index of $\mathbb{Z}(g_1)_q + \cdots + \mathbb{Z}(g_n)_q$ in the division closure of $\mathbb{Z}(g_1)_q + \cdots + \mathbb{Z}(g_n)_q$ in $\tilde{A}_q(k)$, which is finite as $\tilde{A}_q(k)$ is finitely generated by the Mordell-Weil theorem.

Define $\hat{N}(p_q)$ as the smallest natural number such that $\hat{N}(p_q)p_q \in \tilde{A}_q(k) + (\tilde{A}_q)_{\text{tors}}$. Thus, we have that $\hat{N}(p_q)p_q = p'_q + p''_q$ for some $p'_q \in \tilde{A}_q(k)$ and some $p''_q \in (\tilde{A}_q)_{\text{tors}}$. We deduce from Masser's upper bound for the order of a torsion point in [95] that the order of p''_q is bounded polynomially in terms of $[k(p'_q, p_q) : k] = [k(p_q) : k] \leq [F(p) : F]$ with an exponent that depends only on $\dim \tilde{A}_q$ and a constant that depends on $\tilde{A}_q$ and k, but is independent of p. It follows that $\tilde{N}(p_q)$ is bounded by the product of this bound and $\hat{N}(p_q)$.

It remains to bound $\hat{N}(p_q)$. We know that there is $p'''_q \in (\tilde{A}_q)_{\text{tors}}$ such that $\hat{N}(p_q)(p_q + p'''_q) = p'_q$. It follows from the minimality of $\hat{N}(p_q)$ that p'_q is not divisible in $\tilde{A}_q(k)$ by any prime dividing $\hat{N}(p_q)$. Furthermore, p'_q is of infinite order since p is and the specialization homomorphism $\mathbb{Z}g_1 + \cdots + \mathbb{Z}g_n \to \mathbb{Z}(g_1)_q + \cdots + \mathbb{Z}(g_n)_q$ is injective. By Corollary 2.1.5 in [110], $\hat{N}(p_q)$ can be bounded from above by

$$[k(p_q + p'''_q, (\tilde{A}_q)_{\text{tors}}) : k((\tilde{A}_q)_{\text{tors}})] =$$
$$[k(p_q, (\tilde{A}_q)_{\text{tors}}) : k((\tilde{A}_q)_{\text{tors}})] \leq [k(p_q) : k] \leq [F(p) : F]$$

times a constant that depends only on $\tilde{A}_q$ and k and we are done. □

5.2. 2-adic impossible intersections with isogeny orbits

This section is inspired by the work of Stoll [175] and Mavraki [109]. Our proof of Theorem 5.2.1 crucially uses a result of theirs.

Let $\bar{\mathbb{Q}}_2$ denote a fixed algebraic closure of the field of 2-adic numbers $\mathbb{Q}_2$ and let $\mathbb{C}_2$ denote a fixed completion of $\bar{\mathbb{Q}}_2$ with respect to the absolute value on $\bar{\mathbb{Q}}_2$ that extends the 2-adic absolute value on $\mathbb{Q}_2$. This absolute value extends to an absolute value on $\mathbb{C}_2$ that we will denote by $|\cdot|$. All varieties in this section are defined over the field $\mathbb{C}_2$.

Let $Y(2) = \mathbb{A}^1 \backslash \{0, 1\}$ and let $\mathcal{E} \hookrightarrow Y(2) \times \mathbb{P}^2$ be the Legendre family of elliptic curves defined by the equation $Y^2Z = X(X - Z)(X - \lambda Z)$, where λ is the affine coordinate on $Y(2)$ and $[X : Y : Z]$ are homogeneous projective coordinates on $\mathbb{P}^2$. There is a natural surjective morphism $\mathcal{E} \to Y(2)$. We denote the fiber of $\mathcal{E}$ over $\lambda \in Y(2)(\mathbb{C}_2)$ by $\mathcal{E}_\lambda$ and identify it with its image under the projection to $\mathbb{P}^2$. Associated to the elliptic curve $\mathcal{E}_\lambda$ is its j-invariant that we denote by $j(\mathcal{E}_\lambda) \in \mathbb{C}_2$.

THEOREM 5.2.1. *Suppose that* $\lambda_0, \alpha \in \mathbb{C}_2\backslash\{0,1\}$ *satisfy* $|\alpha| = |\alpha - 1| = 1$ *and* $|j(\mathcal{E}_{\lambda_0})| > 1$. *Then there exists no* $\lambda \in Y(2)(\mathbb{C}_2)$ *such that* $\mathcal{E}_\lambda$ *is isogenous to* $\mathcal{E}_{\lambda_0}$ *and the point* $[\alpha : \sqrt{\alpha(\alpha-1)(\alpha-\lambda)} : 1]$ *is torsion on the elliptic curve* $\mathcal{E}_\lambda$ *(for an arbitrary choice of square root).*

PROOF. Suppose to the contrary that there exists such a $\lambda \in Y(2)(\mathbb{C}_2) = \mathbb{C}_2\backslash\{0,1\}$. Since the point $[\alpha : \sqrt{\alpha(\alpha-1)(\alpha-\lambda)} : 1]$ is torsion on the elliptic curve $\mathcal{E}_\lambda$ and $|\alpha| = 1$, we have either $\lambda = \alpha$ or $|\alpha^2 - \lambda| < 1$ by Mavraki's Theorem 4.2 in [109] (first proven by Stoll as Theorem 3 in [175]). It follows in both cases that $|\lambda| = |\lambda - 1| = 1$ and hence

$$|j(\mathcal{E}_\lambda)| = \left|2^8 \frac{(\lambda^2-\lambda+1)^3}{\lambda^2(\lambda-1)^2}\right| < 1.$$

If now additionally $\mathcal{E}_\lambda$ is isogenous to $\mathcal{E}_{\lambda_0}$, it follows that $j(\mathcal{E}_{\lambda_0})$ is a zero of the monic polynomial $P(T) = \Phi_n(T, j(\mathcal{E}_\lambda)) \in \mathbb{Z}[j(\mathcal{E}_\lambda)][T]$ for some $n \in \mathbb{N}$ (Φ_n the n-th modular polynomial as defined for example in Chapter 5, §2 of [83]). Since $|j(\mathcal{E}_\lambda)| < 1$, all coefficients of $P(T)$ are at most 1 in absolute value. It follows that $|j(\mathcal{E}_{\lambda_0})| \leq 1$, a contradiction. □

5.3. Families of semiabelian varieties with many isogenous fibers

DEFINITION 5.3.1. *Let* K *be a field. A semiabelian scheme* $\mathcal{G}$ *over a* K*-variety* S *is a smooth separated commutative group scheme over* S *with connected fibers such that for every (not necessarily closed) point* $s \in S$ *the fiber* $\mathcal{G}_s$ *is an extension of an abelian variety* A_s *by a torus* T_s: $0 \to T_s \to \mathcal{G}_s \to A_s \to 0$, *where a torus is an algebraic group that is isomorphic to the product of a finite number of copies of* $\mathbb{G}_m$ *over the algebraic closure of its field of definition.*

This definition agrees with the one in [45], but is more restrictive than the one in [25], where disconnected fibers are allowed and separatedness is not required. The fibers of a semiabelian scheme are geometrically connected by Proposition 1.34 in [112]. Note that $\mathcal{G}$ is automatically finitely presented over S (see [174]), but we will not use this fact.

DEFINITION 5.3.2. *A semiabelian variety over a field* K *is a semiabelian scheme of finite type over* $\operatorname{Spec} K$. *A homomorphism of algebraic groups between semiabelian varieties is called an isogeny if it is finite, flat, and surjective.*

By Lemma 1 in Section 7.3 of [25], an isogeny between semiabelian varieties is the same as a homomorphism of algebraic groups with finite kernel between semiabelian varieties of the same dimension. From now on, all varieties in this section will be defined over $\bar{\mathbb{Q}}$, unless explicitly stated otherwise.

The following example shows that one has to be careful with the definition of weakly special curves in semiabelian schemes: Let E_0 be a fixed elliptic curve, defined over $\bar{\mathbb{Q}}$, and let $\mathcal{G} \to E_0$ be the universal family of extensions of E_0 by $\mathbb{G}_m$ (where we identify E_0 with its dual elliptic curve) with zero section $\epsilon : E_0 \to \mathcal{G}$. Consider the semiabelian scheme $\mathcal{G}' = \mathcal{G} \times_{E_0} (E_0 \times E_0) \to E_0$ of relative dimension 3, where the morphism $E_0 \times E_0 \to E_0$ is the projection to the first factor. Let $\Delta \subset E_0 \times E_0$ be any translate of an abelian subvariety of dimension 1 that dominates both factors and put $\mathcal{C} = \epsilon(E_0) \times_{E_0} \Delta$. Then the curve $\mathcal{C}$ potentially intersects the isogeny orbit of a subgroup of finite rank of $\mathcal{G}'_p(\bar{\mathbb{Q}})$ for some $p \in E_0(\bar{\mathbb{Q}})$ in infinitely many points. The reason for this is that $\mathcal{G}_p$ and $\mathcal{G}_q$ ($q \in E_0(\bar{\mathbb{Q}})$) are isogenous if $mp = nq$ for some $m, n \in \mathbb{Z}\backslash\{0\}$. (See Lemma 1 in [79] and the explanations on pp. 7–8 of that same article.)

If one wanted to prove an analogue of Theorem 3.1.3 for semiabelian schemes, one would have to take into account this kind of weakly special curves, which does not have a natural analogue in the world of abelian schemes, but is also covered by the general definition of weakly special subvarieties of connected mixed Shimura varieties (Definition 4.1 in [140]).

Another kind of weakly special curves without a natural analogue in the abelian case are the so-called Ribet curves constructed by Bertrand. They are even special. See [17] and [18] for details. Bertrand's construction inspired us to look for other types of weakly special curves without a natural analogue in the abelian case.

We will not prove an analogue of Theorem 3.1.3, but do something different instead: The following theorem characterizes semiabelian schemes of finite type over a curve with infinitely many pairwise isogenous fibers. In the abelian case, this has been done by Orr over the field of complex numbers in [124]. We denote the coarse moduli space of principally polarized abelian varieties of dimension g by A_g. Recall that for an arbitrary abelian variety A we denote its dual abelian variety by $\hat{A}$.

THEOREM 5.3.3. *Let S be a smooth irreducible curve with generic point ξ and let $\mathcal{G} \to S$ be a semiabelian scheme of finite type. Let G_0 be a fixed semiabelian variety. Suppose that there exist infinitely many $s \in S(\bar{\mathbb{Q}})$ such that the fiber $\mathcal{G}_s$ is isogenous to G_0. Then the base change of the generic fiber $\mathcal{G}_\xi$ to $\overline{\bar{\mathbb{Q}}(S)}$ admits*

(1) either a flat surjective homomorphism of algebraic groups with affine kernel to an abelian variety A that is defined over $\bar{\mathbb{Q}}$,

(2) or an isogeny to the product of a semiabelian variety that is defined over $\bar{\mathbb{Q}}$ with an abelian variety B.

In case (1), the point in $\hat{A}^r\left(\overline{\bar{\mathbb{Q}}(S)}\right)$ ($r \in \mathbb{N} \cup \{0\}$) that is associated by the Weil-Barsotti formula to the base change of $\mathcal{G}_\xi$ to $\overline{\bar{\mathbb{Q}}(S)}$ lies in a translate of an abelian subvariety of $\hat{A}^r$ of dimension at most 1 that is defined over $\bar{\mathbb{Q}}$.

In case (2), the point in $A_g\left(\overline{\bar{\mathbb{Q}}(S)}\right)$ that is associated to some choice of principally polarized abelian variety B' isogenous to B lies in a weakly special subvariety of A_g of dimension at most 1 that is defined over $\bar{\mathbb{Q}}$.

PROOF. The semiabelian variety G_0 sits in an exact sequence

$$0 \to \mathbb{G}_m^r \to G_0 \to A_0 \to 0$$

for some abelian variety A_0 and some non-negative integer r. By the Weil-Barsotti formula (Proposition 3.1.5.1 in [80] in the case $S = \operatorname{Spec} \bar{\mathbb{Q}}$), this extension corresponds to a point γ in $\hat{A_0}^r(\bar{\mathbb{Q}})$.

The generic fiber $\mathcal{G}_\xi$ sits inside an exact sequence $0 \to T \to \mathcal{G}_\xi \to A \to 0$ with a torus T and an abelian variety A over $\bar{\mathbb{Q}}(S)$. We can assume that $T = \mathbb{G}^{r'}_{m,\bar{\mathbb{Q}}(S)}$ for a non-negative integer r' after a finite surjective base change $S' \to S$ and replacing S by S'.

By "spreading out" (see Theorem 3.2.1 and Table 1 on pp. 306–307 in [144]), there exists an open subset U of S such that A extends to an abelian scheme $\mathcal{A}$ over U, the homomorphism of algebraic groups $\mathcal{G}_\xi \to A$ extends to a flat and surjective homomorphism of group schemes $\mathcal{G}_U := \mathcal{G} \times_S U \to \mathcal{A}$, and the isomorphism of algebraic groups between $T = \mathbb{G}^{r'}_{m,\bar{\mathbb{Q}}(S)}$ and the kernel of $\mathcal{G}_\xi \to A$ extends to an isomorphism of group schemes between $\mathbb{G}^{r'}_{m,U}$ and the kernel of $\mathcal{G}_U \to \mathcal{A}$.

We can assume without loss of generality that $S = U$. Since there exist infinitely many $s \in S(\bar{\mathbb{Q}})$ such that G_0 is isogenous to $\mathcal{G}_s$, we can deduce that $r = r'$.

By Proposition 3.1.5.1 in [80], the extension $0 \to \mathbb{G}^r_{m,S} \to \mathcal{G} \to \mathcal{A} \to 0$ corresponds to a homomorphism from the constant étale sheaf of groups over S associated to $\mathbb{Z}^r$ into the étale sheaf of groups over S associated to the dual abelian scheme $\hat{\mathcal{A}}$ of $\mathcal{A}$. Such a homomorphism corresponds in turn to a section $\zeta : S \to \hat{\mathcal{A}}^{(r)}$, where $\hat{\mathcal{A}}^{(r)}$ is the r-th fibered power of $\hat{\mathcal{A}}$ over S. For each $s \in S(\bar{\mathbb{Q}})$, the extension $0 \to \mathbb{G}^r_m \to \mathcal{G}_s \to \mathcal{A}_s \to 0$ corresponds to $\zeta(s) \in \hat{\mathcal{A}}^r_s(\bar{\mathbb{Q}})$.

By Lemma 1 in [79] and the explanations on pp. 7–8 of that same article, we see that if G_0 and $\mathcal{G}_s$ are isogenous for $s \in S(\bar{\mathbb{Q}})$ then there exists an isogeny $\phi : A_0 \to \mathcal{A}_s$ and a matrix $\Phi \in M_r(\mathbb{Z})$ with non-zero determinant such that $\hat{\phi}^r(\zeta(s)) = \Phi\gamma$ for the power of the dual isogeny $\hat{\phi}^r : \hat{\mathcal{A}_s}^r \to \hat{A_0}^r$, where $M_r(\mathbb{Z})$ acts on $\hat{A_0}^r(\bar{\mathbb{Q}})$ in the natural way.

This implies that $(\deg \phi)\zeta(s) = \left(\hat{\tilde{\phi}}^r \circ \Phi\right)(\gamma)$, where $\tilde{\phi} : \mathcal{A}_s \to A_0$ is the isogeny such that $\tilde{\phi} \circ \phi$ is equal to multiplication by $\deg \phi$ on A_0 and $\hat{\tilde{\phi}}^r$ is the r-th power of its dual isogeny. Since there are infinitely many $s \in S(\bar{\mathbb{Q}})$ such that $\mathcal{G}_s$ is isogenous to G_0, we can deduce that the curve $\zeta(S) \subset \hat{\mathcal{A}}^{(r)}$ contains infinitely many points of the isogeny orbit of the subgroup of rank ≤ 1 of all division points of multiples of γ in $\hat{A_0}^r(\bar{\mathbb{Q}})$, where we use the terminology of Chapters 3 and 4.

If $\mathcal{A}$ is isotrivial, then the first part of assertion (1) is automatically satisfied and its second part follows from the Mordell-Lang conjecture for a curve in an abelian variety over $\bar{\mathbb{Q}}$ (which is a theorem due to Faltings [42] and Raynaud [146]) and we are done. So we can assume that $\mathcal{A}$ is not isotrivial.

Since $\mathcal{A}$ is not isotrivial, it now follows from Theorem 3.1.3 that some non-zero integral multiple $\tilde{\gamma}$ of the value of ζ at the geometric generic point of S is equal to the base change of a $\bar{\mathbb{Q}}$-point of the $\overline{\bar{\mathbb{Q}}(S)}/\bar{\mathbb{Q}}$-trace of the base change of $\hat{\mathcal{A}}^r_\xi$ to $\overline{\bar{\mathbb{Q}}(S)}$. Here, we use the $\overline{\bar{\mathbb{Q}}(S)}/\bar{\mathbb{Q}}$-trace and -image as defined in Chapter VIII, §§1 and 3 of [81]. We let N denote a non-zero integer such that N times the value of ζ at the geometric generic point of S is equal to $\tilde{\gamma}$.

The base change of $\mathcal{A}_\xi$ to $\overline{\bar{\mathbb{Q}}(S)}$ is isogenous to the product of the base change I of its $\overline{\bar{\mathbb{Q}}(S)}/\bar{\mathbb{Q}}$-image with another abelian variety B. We obtain an isogeny $\psi : (\mathcal{A}_\xi)_{\overline{\bar{\mathbb{Q}}(S)}} \to I \times B$, where we use subscripts to denote base changes and the morphism to the base change of the image can be taken to be the natural one.

Taking the $\overline{\bar{\mathbb{Q}}(S)}/\bar{\mathbb{Q}}$-trace or -image commutes with products as can be checked directly from the defining universal property. Furthermore, the dual of the image is the trace of the dual by Theorem 6.2 in [33]. Therefore, we can identify $\hat{I}^r$ with the base change to $\overline{\bar{\mathbb{Q}}(S)}$ of the $\overline{\bar{\mathbb{Q}}(S)}/\bar{\mathbb{Q}}$-trace of $\left(\hat{\mathcal{A}}^r_\xi\right)_{\overline{\bar{\mathbb{Q}}(S)}}$. Let $\tilde{G}$ be the semiabelian variety with quotient abelian variety $I \times B$ that corresponds via the Weil-Barsotti formula to $(\tilde{\gamma}, 0_{\hat{B}^r}) \in \hat{I}^r\left(\overline{\bar{\mathbb{Q}}(S)}\right) \times \hat{B}^r\left(\overline{\bar{\mathbb{Q}}(S)}\right) = \widehat{I \times B}^r\left(\overline{\bar{\mathbb{Q}}(S)}\right)$.

Let G' be the semiabelian variety with quotient abelian variety I that corresponds to $\tilde{\gamma}$ via the Weil-Barsotti formula. Then $\tilde{G}$ is isomorphic to $G' \times B$. As I is defined over $\bar{\mathbb{Q}}$ and $\tilde{\gamma}$ is the base change of a $\bar{\mathbb{Q}}$-point, the semiabelian variety G' is defined over $\bar{\mathbb{Q}}$ as well.

Thanks to Lemma 1 in [79], the isogeny ψ and the isogeny $\mathbb{G}_m^r \to \mathbb{G}_m^r$ given by raising each coordinate to the N-th power patch together to an isogeny between the geometric generic fiber of $\mathcal{G}$ and $\tilde{G}$. Therefore, the first part of assertion (2) holds.

Let B' be a principally polarized abelian variety that is isogenous to B. After replacing S by an open subset of a finite cover and $\mathcal{A}$ by its corresponding pullback, we can assume that B' and its principal polarization, B, I as well as the isogenies $(\mathcal{A}_\xi)_{\overline{\bar{\mathbb{Q}}(S)}} \to I \times B$ and $B \to B'$ are defined over $\bar{\mathbb{Q}}(S)$ and spread out over S. Since the quotient abelian varieties of isogenous semiabelian varieties are necessarily isogenous to each other (see Lemma 1 and the following remarks in [79]), the second part of assertion (2) then follows from Orr's Theorem 1.2 in [124] and the theorem is proven. □

5.4. Using the Galois operation on torsion points

Let S be a geometrically irreducible affine variety, defined over a number field $K \subset \bar{\mathbb{Q}}$, and let ξ denote the generic point of S. Let $\pi : \mathcal{A} \to S$ be a principally polarized abelian scheme of relative dimension g over S, also defined over K. We denote the zero section of $\mathcal{A}$ by ϵ. In this section, we identify all varieties over K with their base change to $\bar{\mathbb{Q}}$ and "irreducible" will always mean "geometrically irreducible" when the base field is contained in $\bar{\mathbb{Q}}$. In particular, a homomorphism between two abelian varieties, both defined over some number field, is not assumed to be defined over the ground field and two abelian varieties, both defined over some number field, are called isogenous if they are isogenous over $\bar{\mathbb{Q}}$.

By Proposition 6.10 in [120], the double of the principal polarization λ on $\mathcal{A} \to S$ is induced by the line bundle $\mathcal{L} = (\mathrm{id}_{\mathcal{A}}, \lambda)^*\mathcal{P}$ on $\mathcal{A}$, where $\mathcal{P}$ is the Poincaré line bundle on $\mathcal{A} \times_S \hat{\mathcal{A}}$ and $\hat{\mathcal{A}}$ is the dual abelian scheme of $\mathcal{A}$. In Proposition 6.10 in [120], the abelian scheme $\mathcal{A}$ is assumed to be projective over S; this assumption is however unnecessary as it is only used to ensure that the dual abelian scheme $\hat{\mathcal{A}}$ exists, which is guaranteed by Theorem 1.9 in Chapter I of [45].

The restriction of $\mathcal{L}$ to each fiber of $\mathcal{A} \to S$ is symmetric by Theorem 8.8.4 in [20]. The restrictions are also ample as 2λ is a polarization and ampleness is preserved under algebraic equivalence. By Théorème 4.7.1 in [63] and Proposition 13.63 in [58], the line bundle $\mathcal{L}$ is relatively ample for π as defined in Definition 13.60 in [58].

Since S is affine, the line bundle $\mathcal{L}$ is ample. Thanks to Theorem II.7.6 in [71], there exists an immersion $\mathcal{A} \hookrightarrow \mathbb{P}^{R_2} \times S$ associated to the l-th tensor power of $\mathcal{L}$, all defined over K, for some $l \in \mathbb{N}$ large enough and some $R_2 \in \mathbb{N}$. As this immersion is proper, it is actually a closed embedding by [178], Tag 01IQ. In particular, $\mathcal{A}$ is projective over S. We can choose l to be divisible by 6.

For $s \in S$, we denote by $\mathcal{A}_s$ the fiber of $\mathcal{A}$ over s and by $\mathcal{L}_s$ the restriction of $\mathcal{L}$ to $\mathcal{A}_s$. The morphism $\mathcal{A} \to S$ is flat and projective. Therefore, the invertible sheaf $\mathcal{L}^{\otimes l}$ on $\mathcal{A}$ is flat over S and we can apply Theorem III.12.8 and Corollary III.12.9 from [71]. They show that there exists $U \subset S$ open and Zariski dense and a non-negative integer r such that the function $s \mapsto \dim_{\bar{\mathbb{Q}}(s)} H^0(\mathcal{A}_s, \mathcal{L}_s^{\otimes l})$ is constant on U and $(\pi|_{\mathcal{A}_U})_* \left((\mathcal{L}^{\otimes l})|_{\mathcal{A}_U}\right) \simeq \mathcal{O}_U^{\oplus r}$, where $\mathcal{A}_U = \mathcal{A} \times_S U$. We can choose U to be affine and defined over K. We replace $\mathcal{A} \to S$ by $\mathcal{A} \times_S U \to U$.

It follows from Corollary III.12.9 in [71] that we can choose an $\mathcal{O}_S(S)$-basis of $H^0(\mathcal{A}, \mathcal{L}^{\otimes l}) = H^0\left(S, \pi_*\left(\mathcal{L}^{\otimes l}\right)\right) \simeq \mathcal{O}_S(S)^{\oplus r}$ that restricts to a $\bar{\mathbb{Q}}$-basis of $H^0(\mathcal{A}_s, \mathcal{L}_s^{\otimes l})$ for each $s \in S(\bar{\mathbb{Q}})$. We can choose the embedding $\mathcal{A} \hookrightarrow \mathbb{P}^{R_2} \times S$ such that the homogeneous coordinates on $\mathbb{P}^{R_2}$ pull back to the elements of this basis. In particular, the map $H^0(\mathbb{P}^{R_2}, \mathcal{O}_{\mathbb{P}^{R_2}}(1)) \to H^0(\mathcal{A}_s, \mathcal{L}_s^{\otimes l})$ is surjective for all $s \in S(\bar{\mathbb{Q}})$.

Since S is affine, there is a closed embedding $S \hookrightarrow \mathbb{A}^{R_1}$, defined over K, for some $R_1 \in \mathbb{N}$. By composing with the open immersion $\mathbb{A}^{R_1} \hookrightarrow \mathbb{P}^{R_1}$ and the Segre embedding, we obtain an immersion $\mathcal{A} \hookrightarrow \mathbb{P}^{R_1 R_2 + R_1 + R_2}$, also defined over K. The degree deg of a subvariety of $\mathcal{A}$ is defined to be the projective degree of the Zariski closure of its image under this immersion. Suppose that the natural morphism $\rho : S \to A_g$ to the coarse moduli space of principally polarized abelian varieties of dimension g, which is defined over K, is quasi-finite with fibers of cardinality at most M_1.

We include the immersion and the morphism ρ in the data associated to $\mathcal{A}$ so that constants depending on $\mathcal{A}$ are also allowed to depend on the choice of immersion and on M_1. Let A_0 be a fixed abelian variety of dimension g, defined over K.

THEOREM 5.4.1. *Let $\mathcal{V} \subset \mathcal{A}$ be an irreducible subvariety, defined over K, such that $\pi(\mathcal{V}) = S$. Suppose that the set of $x \in \mathcal{V}(\bar{\mathbb{Q}})$ such that x is a torsion point of the fiber $\mathcal{A}_{\pi(x)}$ and such that $\mathcal{A}_{\pi(x)}$ is isogenous to A_0 is Zariski dense in $\mathcal{V}$. Then $\mathcal{V}_\xi$ is equal to a union of translates of abelian subvarieties of $\mathcal{A}_\xi$ by torsion points of $\mathcal{A}_\xi$ (over $\overline{\bar{\mathbb{Q}}(S)}$).*

For Theorem 5.4.1 to hold, some condition on the morphism ρ is clearly necessary. For example, if $\mathcal{A} \to S$ is an abelian scheme of positive relative dimension with a Zariski dense set of pairwise isogenous fibers, then the image of the diagonal section of the abelian scheme $\mathcal{A} \times_S \mathcal{A} \to \mathcal{A}$ contains a Zariski dense set of torsion points that lie on pairwise isogenous fibers. However, the geometric generic fiber of the diagonal section is not a torsion point.

Theorem 5.4.1 also holds if ρ is just assumed to be generically finite and we drop the assumption that S is affine since we can replace S by an open affine Zariski dense subset of S, defined over K, restricted to which ρ is quasi-finite.

Theorem 5.4.1 will follow from the following proposition together with Lemma 5.4.4. The method of the proof of the proposition is the same as in the article [72] by Hindry and goes back to Lang [82].

PROPOSITION 5.4.2. *Let $\mathcal{V} \subset \mathcal{A}$ be an irreducible subvariety, defined over K. Suppose that $x \in \mathcal{V}(\bar{\mathbb{Q}})$ is a torsion point of the fiber $\mathcal{A}_{\pi(x)}$ and that $\mathcal{A}_{\pi(x)}$ is isogenous to A_0. Then one of the following two possibilities holds:*

1. *x lies in a translate of a positive-dimensional abelian subvariety of $\mathcal{A}_{\pi(x)}$ that is contained in $\mathcal{V}_{\pi(x)}$, or*
2. *the order of x is bounded by a constant that depends only on A_0, K, $\mathcal{A}$, $\dim \mathcal{V}$, and $\deg \mathcal{V}$.*

We will use the following lemma to prove Proposition 5.4.2. It can be regarded as a uniform version within an isogeny class of a theorem of Serre.

LEMMA 5.4.3. *Let $\mathcal{V} \subset \mathcal{A}$ be an irreducible subvariety, defined over K. There exists a constant $B \in \mathbb{N}$, depending only on A_0, K, $\mathcal{A}$, $\dim \mathcal{V}$, and $\deg \mathcal{V}$, such that for all $a, M \in \mathbb{N}$ with $\gcd(a, M) = 1$ there exists $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/K)$ with the following property: For all torsion points $x \in \mathcal{A}(\bar{\mathbb{Q}})$ of order M such that $\mathcal{A}_{\pi(x)}$ is isogenous to A_0, we have $\sigma(\pi(x)) = \pi(x)$, $\sigma(x) = a^B x$, and σ fixes every irreducible component of $\mathcal{V}_{\pi(x)}$.*

PROOF. (of Lemma 5.4.3) Let $x \in \mathcal{A}(\bar{\mathbb{Q}})$ be a torsion point of order M such that the fiber $\mathcal{A}_{\pi(x)}$ is isogenous to A_0. Let $\phi : A_0 \to \mathcal{A}_{\pi(x)}$ be an isogeny and let $y \in A_0(\bar{\mathbb{Q}})$ be a torsion point such that $\phi(y) = x$. Choose $N \in \mathbb{N}$ large enough so that y belongs to $A_0[N]$, the set of torsion points of A_0 of order dividing N, and $\ker \phi \subset A_0[N]$. The order M of x is equal to the greatest common divisor of all $n \in \mathbb{N}$ with $ny \in \ker \phi$. In particular, M divides N.

Let $a \in \mathbb{N}$ with $\gcd(a, M) = 1$ be given and fix $\hat{a} \in \hat{\mathbb{Z}}^*$, independently of N, such that $\hat{a} \equiv a \mod M$. By Théorème 3 in [192], due to Serre ([167], No. 136, Théorème 2'), there exists a constant $c = c(A_0, K) \in \mathbb{N}$ such that there exists $\sigma_a \in \mathrm{Gal}(\bar{\mathbb{Q}}/K)$ acting on $\varprojlim A_0[n] \simeq \hat{\mathbb{Z}}^{2g}$ as multiplication by $\hat{a}^c$. We now choose $\tilde{a} \in \mathbb{N}$ such that $\tilde{a} \equiv \hat{a} \mod N$ and replace a by $\tilde{a}$. The Galois automorphism σ will not depend on the choice of $\tilde{a}$, but only on σ_a, $\mathcal{A}$, and $\deg \mathcal{V}$. We deduce that $\sigma_a(y) = a^{c(A_0,K)} y$.

If $\lambda_s : \mathcal{A}_s \to \widehat{\mathcal{A}_s}$ denotes the principal polarization on $\mathcal{A}_s$ ($s \in S$), then we have that $\hat{\phi} \circ \lambda_{\pi(x)} \circ \phi$ is a polarization of A_0. By Théorème 1.2 in [159], every homomorphism between A_0 and $\widehat{A_0}$ is defined over a Galois extension of K of degree at most $F(g) = 4\beta(g)6^{2(g-1)}(g!)^2$, where $\beta(g) = 3$ if $g \notin \{2, 4, 6\}$, while $\beta(2) = 8$, $\beta(4) = 75$, and $\beta(6) = \frac{49}{12}$. Hence, the polarization $\hat{\phi} \circ \lambda_{\pi(x)} \circ \phi$ of A_0 is fixed by $\sigma_a^{\circ F(g)!}$.

Since σ_a acts on $A_0[N] \supset \ker \phi$ as multiplication by a constant that is coprime to N, we know that $\sigma_a^{\circ F(g)!}(\ker \phi) = \ker \phi$. Therefore, there is an isomorphism $\iota : \mathcal{A}_{\pi(x)} \to \mathcal{A}_{\sigma_a^{\circ F(g)!}(\pi(x))}$ such that $\sigma_a^{\circ F(g)!}(\phi) = \iota \circ \phi$ (see Theorem 5.13 in [112]).

Since $\hat{\phi} \circ \lambda_{\pi(x)} \circ \phi$ is fixed by $\sigma_a^{\circ F(g)!}$ and the polarization of $\mathcal{A}$ is defined over K, we deduce that

$$\hat{\phi} \circ \lambda_{\pi(x)} \circ \phi = \widehat{\sigma_a^{\circ F(g)!}(\phi)} \circ \lambda_{\sigma_a^{\circ F(g)!}(\pi(x))} \circ \sigma_a^{\circ F(g)!}(\phi) = \hat{\phi} \circ \hat{\iota} \circ \lambda_{\sigma_a^{\circ F(g)!}(\pi(x))} \circ \iota \circ \phi.$$

Note that $\widehat{\sigma_a^{\circ F(g)!}(\phi)} = \sigma_a^{\circ F(g)!}\left(\hat{\phi}\right)$ since dualizing commutes with extending the base field. As ϕ and $\hat{\phi}$ are isogenies, it follows that $\lambda_{\pi(x)} = \hat{\iota} \circ \lambda_{\sigma_a^{\circ F(g)!}(\pi(x))} \circ \iota$, so $\mathcal{A}_{\pi(x)}$ and $\mathcal{A}_{\sigma_a^{\circ F(g)!}(\pi(x))}$ are isomorphic as polarized abelian varieties.

Hence, the point $\rho(\pi(x))$ is fixed by $\sigma_a^{\circ F(g)!}$. It follows that the finite set $\rho^{-1}(\rho(\pi(x)))$ of cardinality at most M_1 is permuted by $\sigma_a^{\circ F(g)!}$ and therefore the Galois automorphism $\sigma_a^{\circ F(g)! M_1!}$ fixes $\pi(x)$.

By Proposition 2.1 in [43], the Zariski closure of $\mathcal{V}$ is cut out in $\mathbb{P}^{R_1 R_2 + R_1 + R_2}$ by homogeneous forms of degree at most $\deg \mathcal{V}$. By specialization, it follows that

$\mathcal{V}_{\pi(x)}$ is cut out in $\mathbb{P}^{R_2}$ by forms of degree at most $\deg \mathcal{V}$. The number of irreducible components of $\mathcal{V}_{\pi(x)}$ is certainly bounded from above by the degree of $\mathcal{V}_{\pi(x)}$ (as a subvariety of $\mathbb{P}^{R_2}$), which in turn is bounded from above by $M_2 = (\deg \mathcal{V})^{R_2}$ by Proposition 3.3 in [130] (a version of Bézout's theorem).

Since $\mathcal{V}$ is defined over K, these irreducible components are permuted by the Galois automorphism $\sigma_a^{\circ F(g)!M_1!}$ and therefore fixed by $\sigma_a^{\circ F(g)!M_1!M_2!}$.

By Théorème 1.2 in [159], the isogeny ϕ is defined over a Galois extension of $K(\pi(x))$ of degree at most $F(g)$ with $F(g)$ as above. We deduce that $\sigma = \sigma_a^{\circ M_1!M_2!(F(g)!)^2}$ fixes ϕ and

$$a^B x = \phi(a^B y) = \phi\left(\sigma(y)\right) = \sigma(\phi(y)) = \sigma(x)$$

with $B = c(A_0, K)M_1!M_2!(F(g)!)^2$. $\square$

Proof. (of Proposition 5.4.2) Let $x \in \mathcal{V}(\bar{\mathbb{Q}})$ be a torsion point of order M such that the fiber $\mathcal{A}_{\pi(x)}$ is isogenous to A_0. Let X be an irreducible component of $\mathcal{V}_{\pi(x)}$, containing x, and set $m = \dim X \leq \dim \mathcal{V}$.

It follows from Lemma 5.4.3 that $a^B x$ is a Galois conjugate of x under a Galois automorphism that fixes X for all $a \in \mathbb{N}$ with $\gcd(a, M) = 1$. By Proposition 2.1 in [43], the Zariski closure of $\mathcal{V}$ in $\mathbb{P}^{R_1R_2+R_1+R_2}$ is cut out by homogeneous forms of degree at most $\deg \mathcal{V}$. By specialization, it follows that $\mathcal{V}_{\pi(x)}$ is cut out in $\mathbb{P}^{R_2}$ by forms of degree at most $\deg \mathcal{V}$. By Proposition 3.3 in [130], the degree $\deg X$ of X as a subvariety of $\mathbb{P}^{R_2}$ is bounded from above by $(\deg \mathcal{V})^{R_2}$. By Proposition 2.1 in [43], X is cut out in $\mathbb{P}^{R_2}$ by forms of degree at most $(\deg \mathcal{V})^{R_2}$.

Suppose that x does not satisfy condition (1) of Proposition 5.4.2. We want to show that it satisfies condition (2).

By Proposition 2 in [72], there is an integer h, depending only on m, such that $\{x\}$ is an irreducible component of $Z = \bigcap_{j=0}^{h} \left[a^{Bj}\right]^{-1}(X)$, where $\left[a^{Bj}\right] : \mathcal{A}_{\pi(x)} \to \mathcal{A}_{\pi(x)}$ denotes the multiplication-by-a^{Bj} morphism, provided that

$$a^{2B} > 2^m \left(\deg \mathcal{V}\right)^{R_2(m+1)} \geq \left(\deg X\right)\left(2(\deg \mathcal{V})^{R_2}\right)^m. \tag{5.4.1}$$

Note that X is cut out in $\mathbb{P}^{R_2}$ by forms of degree at most $(\deg \mathcal{V})^{R_2}$ and that the right-hand side of the inequality is equal to the polynomial $H(X; \cdot)$ for $X \subset \mathbb{P}^{R_2}$ (as defined on p. 579 of [72]) evaluated at $2(\deg \mathcal{V})^{R_2}$. We can deduce the same for any Galois conjugate of x over K that lies in X.

Note that we can apply the results from [72] since our embedding of $\mathcal{A}_{\pi(x)}$ into $\mathbb{P}^{R_2}$ corresponds to an l-th power (with l divisible by 6) of a symmetric ample line bundle and therefore is good as defined on p. 578 of [72]. In particular, the embedding is projectively normal by Theorem 9 in [119] since the map $H^0(\mathbb{P}^{R_2}, \mathcal{O}_{\mathbb{P}^{R_2}}(1)) \to H^0\left(\mathcal{A}_{\pi(x)}, \mathcal{L}_{\pi(x)}^{\otimes l}\right)$ is surjective by construction, and hence Lemmes 3 and 4 from [72] can be applied. In the proof of Proposition 2 in [72], we need furthermore that the Galois automorphism sending x to $a^B x$ fixes the maximal abelian subvariety C of $\mathcal{A}_{\pi(x)}$ with $x + C \subset X$; this is also satisfied since x does not satisfy condition (1) of Proposition 5.4.2 and we therefore have $C = \{\epsilon(\pi(x))\}$.

Letting a vary in Lemma 5.4.3 shows that the number of Galois conjugates of x over K that lie in X is equal to at least $\frac{\phi(M)}{2B^{\omega(M)}}$, where ϕ is Euler's phi function and

$\omega(M)$ denotes the number of distinct prime factors of M. By Théorème 11 in [161], we have $\omega(M) \leq \frac{7 \log M}{5 \log \log M}$ if $M \geq 3$. We can also estimate

$$\phi(M) = M \prod_{p|M} \frac{p-1}{p} \geq M \prod_{j=2}^{\omega(M)+1} \frac{j-1}{j} = \frac{M}{\omega(M)+1} \geq \frac{M}{2\log M + 1}.$$

From now on, we assume that $M \geq 3$. It follows that the number of Galois conjugates of x over K that lie in X is at least

$$\frac{M^{1 - \frac{\log 2}{\log M} - \frac{7 \log B}{5 \log \log M}}}{(2 \log M + 1)}.$$

Let $d = a^B$ for some $a \in \mathbb{N}$ with $\gcd(a, M) = 1$. Since $\deg X \leq (\deg \mathcal{V})^{R_2}$ and X is cut out in $\mathbb{P}^{R_2}$ by forms of degree at most $(\deg \mathcal{V})^{R_2}$, it follows from Lemme 6(iii) in [72] and Proposition 3.3 in [130] that

$$\frac{M^{1 - \frac{\log 2}{\log M} - \frac{7 \log B}{5 \log \log M}}}{(2 \log M + 1)} \leq \deg Z \leq (\deg X) \left(d^{2h} (\deg \mathcal{V})^{R_2} \right)^m \leq d^{2hm} \left(\deg \mathcal{V} \right)^{R_2(m+1)}, \tag{5.4.2}$$

where $\deg Z$ denotes the degree of Z as a subvariety of $\mathbb{P}^{R_2}$.

If M is large enough, there exists a prime number a such that a does not divide M and

$$2^{\frac{m}{2B}} \left(\deg \mathcal{V} \right)^{\frac{R_2(m+1)}{2B}} < a < \left(\frac{M^{1 - \frac{\log 2}{\log M} - \frac{7 \log B}{5 \log \log M}}}{(2 \log M + 1) \left(\deg \mathcal{V} \right)^{R_2(m+1)}} \right)^{\frac{1}{2hmB}}.$$

So (5.4.1) is satisfied and we deduce that (5.4.2) holds. But this contradicts the upper bound on a, so we get an upper bound for M that depends only on A_0, K, $\mathcal{A}$, $\dim \mathcal{V}$, and $\deg \mathcal{V}$. $\square$

LEMMA 5.4.4. *Let $\mathcal{V} \subset \mathcal{A}$ be an irreducible subvariety that dominates S. Suppose that all abelian subvarieties of the base change of $\mathcal{A}_\xi$ to $\overline{\bar{\mathbb{Q}}(S)}$ are defined over $\bar{\mathbb{Q}}(S)$ and that the stabilizer* $\mathrm{Stab}(\mathcal{V}_\xi, \mathcal{A}_\xi)$ *is finite. Then the union of all translates of positive-dimensional abelian subvarieties of $\mathcal{A}_s$ that are contained in $\mathcal{V}_s$ for some $s \in S(\bar{\mathbb{Q}})$ is contained in a proper subvariety of $\mathcal{V}$.*

The proof of Lemma 5.4.4 runs along similar lines as the proof of Lemma 4.3.3. The difference is that the base variety S is now allowed to have dimension bigger than 1. Note that Lemma 5.4.4 could also be obtained as a consequence of the much more general Theorem 12.2 in [52], at least for $\mathcal{A}$ contained in a suitable universal family and then for arbitrary $\mathcal{A}$ as well. However, we again prefer to give a direct proof that does not make use of the language of mixed Shimura varieties.

PROOF. We first pass to a quasi-finite dominant cover $S' \to S$ such that S' is smooth and irreducible and every irreducible component of the base change of $\mathcal{V}_\xi$ to $\overline{\bar{\mathbb{Q}}(S)}$ is defined over $\bar{\mathbb{Q}}(S')$. Set $\mathcal{A}' = \mathcal{A} \times_S S'$. Let $\mathcal{V}'$ be an irreducible component of $\mathcal{V} \times_S S' \hookrightarrow \mathcal{A}'$ that dominates S'.

If η is the generic point of S', then $\mathrm{Stab}(\mathcal{V}'_\eta, \mathcal{A}'_\eta)$ must be finite. Otherwise it would contain a positive-dimensional abelian subvariety of $\mathcal{A}'_\eta$, but as all abelian subvarieties of $\mathcal{A}'_\eta$ are defined over $\bar{\mathbb{Q}}(S)$, this abelian subvariety would be contained in the stabilizer of $\mathcal{V}_\xi$, which could therefore not be finite. Furthermore, $\mathcal{V}'_\eta = \mathcal{V}' \cap \mathcal{A}'_\eta$ is irreducible by Section 2.1.8 of Chapter 0 of [61] and hence geometrically irreducible by our choice of S'.

If the union of all translates of positive-dimensional abelian subvarieties of $\mathcal{A}_s$ that are contained in $\mathcal{V}_s$ for some $s \in S(\bar{\mathbb{Q}})$ is Zariski dense in $\mathcal{V}$, then the union of all translates of positive-dimensional abelian subvarieties of $\mathcal{A}'_s$ that are contained in $\mathcal{V}'_s$ for some $s \in S'(\bar{\mathbb{Q}})$ is Zariski dense in $\mathcal{V}'$. So we can replace $\mathcal{A}$ and $\mathcal{V}$ by $\mathcal{A}'$ and $\mathcal{V}'$ and assume without loss of generality that $\mathcal{V}_\xi$ is geometrically irreducible.

Let $N \in \mathbb{N}$ be a natural number that is larger than the order of $\mathrm{Stab}(\mathcal{V}_\xi, \mathcal{A}_\xi)$. There are finitely many irreducible subvarieties $\mathcal{T}_1, \ldots, \mathcal{T}_R \subset \mathcal{A}$ such that each $\mathcal{T}_i$ dominates S and the union of the $\mathcal{T}_i$ $(i = 1, \ldots, R)$ is equal to the set of points of exact order N on $\mathcal{A}$: First of all, every irreducible component of the pre-image of $\epsilon(S)$ under the multiplication-by-N morphism $[N]$ dominates S by Proposition 2.3.4(iii) in [64] since $[N]$ is étale, so flat (see [111], Proposition 20.7). Therefore, every irreducible component of $[N]^{-1}(\epsilon(S))$ is of dimension $\dim S$. The same holds for any $M \in \mathbb{N}$ that divides N. Furthermore, $[N]^{-1}(\epsilon(S))$ is smooth as $[N]$ is étale and S is smooth. Hence, no two distinct irreducible components of $[N]^{-1}(\epsilon(S))$ intersect each other. So every irreducible component of $[N]^{-1}(\epsilon(S))$ is either contained in $\bigcup_{M|N, M \neq N} [M]^{-1}(\epsilon(S))$ or disjoint from it and our claim follows.

We now consider $\mathcal{W}_i = \mathcal{V} \cap (\mathcal{V} + \mathcal{T}_i)$ for $i \in \{1, \ldots, R\}$. If this variety were equal to $\mathcal{V}$, then we would have $\mathcal{V} \subset \mathcal{V} + \mathcal{T}_i$ and so $\mathcal{V}_\xi \subset \mathcal{V}_\xi + (\mathcal{T}_i)_\xi$. For dimension reasons and thanks to the geometric irreducibility of $\mathcal{V}_\xi$, we would get that $\mathcal{V}_\xi = t + \mathcal{V}_\xi$ for a torsion point $t \in \mathcal{A}_\xi$ of order N. This contradicts our choice of N. So $\mathcal{W}_i \subsetneq \mathcal{V}$.

On the other hand, each positive-dimensional abelian variety contains a point of order N, so the union of all translates of positive-dimensional abelian subvarieties of $\mathcal{A}_s$ that are contained in $\mathcal{V}_s$ for some $s \in S(\bar{\mathbb{Q}})$ is contained in $\bigcup_{i=1}^{R} \mathcal{W}_i$. As every $\mathcal{W}_i$ is a proper closed subset of $\mathcal{V}$ and $\mathcal{V}$ is irreducible, the lemma follows. □

PROOF. (of Theorem 5.4.1) After a quasi-finite dominant base change $S' \to S$ with S' smooth and irreducible and after replacing $\mathcal{A}$ by $\mathcal{A} \times_S S'$ and $\mathcal{V}$ by an irreducible component of $\mathcal{V} \times_S S'$ that dominates S', we can assume that all abelian subvarieties of the base change of $\mathcal{A}_\xi$ to $\overline{\bar{\mathbb{Q}}(S)}$ are defined over $\bar{\mathbb{Q}}(S)$. Here and in the following, it might sometimes be necessary to replace the field of definition K by a finite extension of K and we will do this without further comments. Note that the principal polarization of $\mathcal{A}$ yields a principal polarization of $\mathcal{A} \times_S S'$, that the morphism $S' \to A_g$ factors through $S \to A_g$, and that we can construct a quasi-projective immersion of $\mathcal{A} \times_S S'$ with the same properties as the one of $\mathcal{A}$ (after maybe replacing S' by an open affine Zariski dense subset).

Let A' be the irreducible component of $\mathrm{Stab}(\mathcal{V}_\xi, \mathcal{A}_\xi)$ that contains $0_{\mathcal{A}_\xi}$. Then A' is an abelian subvariety of $\mathcal{A}_\xi$. We can now use the Poincaré reducibility theorem to deduce that there exists another abelian subvariety A'' of $\mathcal{A}_\xi$ such that the natural morphism $A' \times A'' \to \mathcal{A}_\xi$ given by restricting the addition morphism $\mathcal{A}_\xi \times \mathcal{A}_\xi \to \mathcal{A}_\xi$ is an isogeny.

By "spreading out" (see Theorem 3.2.1 and Table 1 on pp. 306–307 in [144]), we can find abelian schemes $\mathcal{A}'$ and $\mathcal{A}''$ over an open Zariski dense subset U of S with generic fibers A' and A'' and a morphism $\alpha : \mathcal{A}' \times_U \mathcal{A}'' \to \mathcal{A} \times_S U$ that extends the isogeny $A' \times A'' \to \mathcal{A}_\xi$. We can assume without loss of generality that $S = U$.

As α is dominant, proper, and maps the image of the zero section to the image of the zero section, it follows that α restricts to an isogeny on each fiber. It suffices to prove that the conclusion of the theorem holds for one of the irreducible components of $\alpha^{-1}(\mathcal{V})$ that dominate $\mathcal{V}$, called $\mathcal{V}'$, inside the family $\mathcal{A}' \times_S \mathcal{A}''$.

By construction, the generic fiber $\mathcal{V}'_\xi$ is equal to $\mathcal{A}'_\xi \times \mathcal{V}''_\xi$, where $\mathcal{V}''$ is the image of $\mathcal{V}'$ under the projection to $\mathcal{A}''$, and hence $\mathcal{V}' = \mathcal{A}' \times_S \mathcal{V}''$. Note that the projection morphism is proper, so $\mathcal{V}''$ is closed in $\mathcal{A}''$.

Let $\epsilon' : S \to \mathcal{A}'$ denote the zero section of $\mathcal{A}'$ and set $\mathcal{V}''' = \alpha(\epsilon'(S) \times_S \mathcal{V}'') \subset \mathcal{A}$. By construction, the stabilizer $\operatorname{Stab}(\mathcal{V}'''_\xi, \mathcal{A}_\xi)$ is finite and the set of torsion points $x \in \mathcal{V}'''(\bar{\mathbb{Q}})$ such that $\mathcal{A}_{\pi(x)}$ is isogenous to A_0 is Zariski dense in $\mathcal{V}'''$.

Combining Proposition 5.4.2 with Lemma 5.4.4 shows that $\mathcal{V}'''$ must be equal to an irreducible component of $[M]^{-1}(\epsilon(S))$ for some $M \in \mathbb{N}$, where $[M] : \mathcal{A} \to \mathcal{A}$ denotes the multiplication-by-M morphism. The theorem follows. □

5.5. Estimating intersection numbers in families of abelian varieties

The goal of this section is to explore under which technical conditions (that we try to choose as generally as possible) an inequality between intersection numbers as in Lemma 4.4.5 can be proven and a similar proof as in Chapter 4 might apply. For this, we have to restrict ourselves to polarized isogenies and put certain technical conditions on the principally polarized abelian variety A_0 and the principally polarized abelian scheme $\mathcal{A}$. We denote by $A_{g,l}$ the fine moduli space of principally polarized abelian varieties of dimension g with symplectic level l-structure (for $l \geq 3$ so that we obtain a fine moduli space). We also use the corresponding universal family $\pi : \mathfrak{A}_{g,l} \to A_{g,l}$. In this section, we consider $A_{g,l}$ as a variety over $\mathbb{C}$.

THEOREM 5.5.1. *Let A and B be abelian varieties of the same dimension g, defined over $\mathbb{C}$. Let $\mathcal{L}$ and $\mathcal{M}$ be line bundles, inducing principal polarizations of A and B respectively. Suppose that every algebraic cycle on A is numerically equivalent to a $\mathbb{Q}$-linear combination of intersections of divisors. Let $\operatorname{End}^s(A)$ and $\operatorname{End}^s(B)$ denote the additive groups of endomorphisms that are fixed by the Rosati involutions associated to $\mathcal{L}$ and $\mathcal{M}$ respectively.*

Let $\phi : B \to A$ be a polarized isogeny with respect to $\mathcal{M}$ and $\mathcal{L}$ and let

$$N = \left[\phi^{-1} \operatorname{End}^s(A)\phi : (\phi^{-1} \operatorname{End}^s(A)\phi) \cap \operatorname{End}^s(B)\right] \in \mathbb{N}.$$

There exists a natural number θ, depending only on A and $\mathcal{L}$, such that for each $k \in \{0, \ldots, g\}$ and each pure-dimensional algebraic cycle U of dimension k on B there exists a pure-dimensional algebraic cycle V of dimension k on A such that $\phi_*(U)$ is numerically equivalent to $\theta^{-1}\left((\deg\phi)^{\frac{1}{g}} N^{-1}\right)^k V$.

PROOF. For a line bundle $\mathcal{N}$ on some abelian variety, we denote by $\phi_\mathcal{N}$ the induced morphism to the dual abelian variety. Since ϕ is polarized, we have

$$n\phi_\mathcal{M} = \phi_{\phi^*\mathcal{L}} = \hat{\phi} \circ \phi_\mathcal{L} \circ \phi$$

for some natural number n. Comparing the degrees of both sides shows that $n^{2g} = (\deg\phi)^2$, so $n = (\deg\phi)^{\frac{1}{g}}$. In particular, $\deg\phi$ is the g-th power of some natural number.

Let now $\mathcal{N}$ be an arbitrary line bundle on A. Then the following equality holds in $\operatorname{Hom}(B, \hat{B}) \otimes_{\mathbb{Z}} \mathbb{Q}$:

$$\phi_{\phi^*\mathcal{N}} = \hat{\phi} \circ \phi_{\mathcal{N}} \circ \phi = \phi_{\phi^*\mathcal{L}} \circ \chi = (\deg \phi)^{\frac{1}{g}} (\phi_{\mathcal{M}} \circ \chi),$$

where

$$\chi = \phi^{-1} \circ \phi_{\mathcal{L}}^{-1} \circ \phi_{\mathcal{N}} \circ \phi \in \phi^{-1} \operatorname{End}^s(A)\phi \subset \operatorname{End}^s(B) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

From the definition of N, we obtain that $N\chi \in \operatorname{End}^s(B)$. It follows from the bijection (induced by the principal polarization $\phi_{\mathcal{M}}$) between line bundles on B up to algebraic equivalence and endomorphisms of B that are fixed by the Rosati involution that there exists a line bundle $\mathcal{N}'$ on B such that $(\phi^*\mathcal{N})^{\otimes N}$ is algebraically equivalent to $\mathcal{N}'^{\otimes (\deg \phi)^{\frac{1}{g}}}$.

We now fix $k \in \{0, \ldots, g\}$ and let U be an arbitrary pure-dimensional algebraic cycle of dimension k on B. Let W be an arbitrary pure-dimensional algebraic cycle of codimension k on A. The additive groups of pure-dimensional algebraic cycles on A of dimension or codimension k respectively modulo numerical equivalence are finitely generated free $\mathbb{Z}$-modules by Example 19.1.3 in [47]. Our hypothesis therefore implies that there exists some natural number M, depending only on A and k, but not on W, such that MW is numerically equivalent to a $\mathbb{Z}$-linear combination of intersections of divisors. It follows from the above that $N^k\phi^*(MW)$ is numerically equivalent to $(\deg \phi)^{\frac{k}{g}} W'$ for some pure-dimensional algebraic cycle W' of codimension k on A.

We compute the intersection product

$$N^k M(\phi_*(U) \cdot W) = \phi_*(U) \cdot (N^k MW) = U \cdot \left(N^k \phi^*(MW)\right) = U \cdot \left((\deg \phi)^{\frac{k}{g}} W'\right).$$

It follows that

$$\phi_*(U) \cdot W \in M^{-1} \left((\deg \phi)^{\frac{1}{g}} N^{-1}\right)^k \mathbb{Z}. \tag{5.5.1}$$

Let d denote the rank of the free $\mathbb{Z}$-module of pure-dimensional algebraic cycles of dimension k on A modulo numerical equivalence. We choose algebraic cycles $V_1, \ldots, V_d$ that yield a basis of this module. It follows from the definition of numerical equivalence (in the form stated in Example 19.1.5(a) in [47]) that d is also the rank of the free $\mathbb{Z}$-module of pure-dimensional algebraic cycles of codimension k on A modulo numerical equivalence and we similarly choose algebraic cycles $W_1, \ldots, W_d$ that yield a basis of that module. Both choices depend only on A and are made independently of U. It follows that $\phi_*(U)$ is numerically equivalent to a linear combination $\sum_{j=1}^d a_j V_j$ with $a_j \in \mathbb{Z}$ $(j = 1, \ldots, d)$. It follows from (5.5.1) that

$$\sum_{j=1}^{d} (W_i \cdot V_j) a_j = \phi_*(U) \cdot W_i \in M^{-1} \left((\deg \phi)^{\frac{1}{g}} N^{-1}\right)^k \mathbb{Z}$$

for $i = 1, \ldots, d$.

The matrix $D = (W_i \cdot V_j)_{i,j=1,\ldots,d}$ with integer entries has non-zero determinant by the definition of numerical equivalence (as in Example 19.1.5(a) in [47]). By multiplying with its adjugate, we find that

$$(\det D) a_j \in M^{-1} \left((\deg \phi)^{\frac{1}{g}} N^{-1}\right)^k \mathbb{Z}$$

for $j = 1, \dots, d$. The theorem follows with θ equal to the least common multiple of all values of $M \det D$ as k ranges over $\{0, \dots, g\}$. $\square$

The condition that every algebraic cycle on A is numerically equivalent to a $\mathbb{Q}$-linear combination of intersections of divisors is satisfied for example if A is simple and g is prime by [176]. (Not every abelian variety satisfies this condition, though; for explicit counterexamples, see [165] and [183].) If A is simple and g is odd and squarefree (e.g. g an odd prime), then it follows from the restrictions in Proposition 5.5.7 in [19] that $\mathrm{End}(A)$ must be commutative.

Corollary 5.5.2. *Let $S \subset A_{g,l}$ be an irreducible subvariety with generic point ξ and set $\mathcal{A} = \mathfrak{A}_{g,l} \times_{A_{g,l}} S$. Let A_0 be a principally polarized abelian variety of dimension g over $\mathbb{C}$ and let $s \in S(\mathbb{C})$ such that there exists a polarized isogeny $\phi : \mathcal{A}_s \to A_0$. If $\mathrm{End}(A_0)$ is commutative, every algebraic cycle on A_0 is numerically equivalent to a $\mathbb{Q}$-linear combination of intersections of divisors, and $\mathrm{End}(A_0) \otimes_{\mathbb{Z}} \mathbb{Q}$ is isomorphic to $\mathrm{End}(\mathcal{A}_\xi) \otimes_{\mathbb{Z}} \mathbb{Q}$, then there exists a natural number θ, depending only on S, A_0, and the choice of principal polarization on A_0, but independent of s and ϕ, such that for each $k \in \{0, \dots, g\}$ and each pure-dimensional algebraic cycle U of dimension k on $\mathcal{A}_s$ there exists a pure-dimensional algebraic cycle V of dimension k on A_0 such that $\phi_*(U) = \theta^{-1}(\deg \phi)^{\frac{k}{g}} V$.*

Here, we denote by $\mathrm{End}(\mathcal{A}_\xi)$ the endomorphism ring of the base change of $\mathcal{A}_\xi$ to an algebraic closure of $\mathbb{C}(S)$. Since $l \geq 3$, this distinction is however irrelevant by Theorem 2.4 in [170].

If $s \in S(\mathbb{C})$, we have a specialization homomorphism $\mathrm{End}(\mathcal{A}_\xi) \to \mathrm{End}(\mathcal{A}_s)$ thanks to “spreading out” (see Theorem 3.2.1 and Table 1 on pp. 306–307 in [144]) and Proposition 2.7 in Chapter I of [45]. For the construction, we might have to pass to the normalization of S, so this specialization homomorphism may not be uniquely determined. It is injective since the kernel of an endomorphism of an abelian scheme of relative dimension g over an irreducible variety is everything if one of its fibers is of dimension g (this follows from the Fiber Dimension Theorem – Corollary 14.116 in [58] –, applied to the image of the endomorphism).

Proof. The corollary follows from Theorem 5.5.1 with $A = A_0$ and $B = \mathcal{A}_s$, once we have shown that the index N in that theorem can be bounded independently of s. As $\mathrm{End}(A_0)$ is commutative, the algebra $\mathrm{End}(\mathcal{A}_s) \otimes_{\mathbb{Z}} \mathbb{Q}$, which is isomorphic to $\mathrm{End}(A_0) \otimes_{\mathbb{Z}} \mathbb{Q}$, is isomorphic to a product of number fields and therefore contains a unique maximal order $\mathfrak{M}$, which contains both $\phi^{-1}\,\mathrm{End}(A_0)\phi$ as well as $\mathrm{End}(\mathcal{A}_s)$.

Consider the natural homomorphism of additive groups

$$\phi^{-1}\,\mathrm{End}^s(A_0)\phi \to \mathfrak{M}/\,\mathrm{End}(\mathcal{A}_s).$$

Its kernel is equal to $(\phi^{-1}\,\mathrm{End}^s(A_0)\phi) \cap \mathrm{End}(\mathcal{A}_s) = (\phi^{-1}\,\mathrm{End}^s(A_0)\phi) \cap \mathrm{End}^s(\mathcal{A}_s)$. It follows that the index N is bounded by the index of $\mathrm{End}(\mathcal{A}_s)$ in $\mathfrak{M}$. By the above, $\mathrm{End}(\mathcal{A}_\xi)$ injects into $\mathrm{End}(\mathcal{A}_s)$ and its image under specialization is an order in $\mathrm{End}(\mathcal{A}_s) \otimes_{\mathbb{Z}} \mathbb{Q}$ since this algebra is isomorphic to $\mathrm{End}(\mathcal{A}_\xi) \otimes_{\mathbb{Z}} \mathbb{Q}$. Therefore, finally, the index N is bounded by the index of $\mathrm{End}(\mathcal{A}_\xi)$ inside the maximal order of $\mathrm{End}(\mathcal{A}_\xi) \otimes_{\mathbb{Z}} \mathbb{Q}$, which depends only on S, but neither on s nor ϕ. $\square$

The next lemma shows that for each principally polarized abelian variety A_0 without CM such that $\mathrm{End}(A_0)$ is commutative and every algebraic cycle on A_0 is numerically equivalent to a $\mathbb{Q}$-linear combination of intersections of divisors, we can find a corresponding $S \subset A_{g,l}$ of positive dimension to which we can apply Corollary 5.5.2. We have to allow S to be of dimension larger than 1, simply because there might not exist any non-isotrivial abelian scheme over a curve with infinitely many fibers isogenous to A_0. This introduces new difficulties if one wants to follow the proof in Chapter 4. However, these difficulties can be avoided if we choose $S \subset A_{g,l}$ weakly special and of positive dimension such that S contains no proper subvariety that is positive-dimensional, weakly special, and contains a point corresponding to an abelian variety that admits a polarized isogeny to A_0. The next lemma shows that this is possible.

LEMMA 5.5.3. *Let A_0 be a principally polarized abelian variety of dimension g over $\mathbb{C}$. If A_0 does not have CM, i.e. its endomorphism algebra does not contain a commutative semi-simple subalgebra of dimension $2g$ over $\mathbb{Q}$, then there exists a positive-dimensional weakly special subvariety $S \subset A_{g,l}$ such that*

- *(i) the generic fiber of the abelian scheme $\mathcal{A} := \mathfrak{A}_{g,l} \times_{A_{g,l}} S \to S$ is an abelian variety with endomorphism algebra isomorphic to the endomorphism algebra of A_0, and*
- *(ii) S contains a Zariski dense set of points $s \in S(\mathbb{C})$ such that there exists a polarized isogeny between $\mathcal{A}_s$ and A_0, and*
- *(iii) there exists no positive-dimensional weakly special subvariety of $A_{g,l}$ that is properly contained in S and contains a complex point s such that there is a polarized isogeny between $\mathcal{A}_s$ and A_0.*

PROOF. We can find a special subvariety S of PEL type of $A_{g,l}$ associated to the endomorphism algebra of A_0 containing a complex point s_0 such that $(\mathfrak{A}_{g,l})_{s_0}$ admits a polarized isogeny to A_0 (cf. Example 3.9 in [114]).

Since A_0 does not have CM, this special subvariety is positive-dimensional. If it properly contains a positive-dimensional weakly special subvariety that contains a complex point s_1 such that there is a polarized isogeny between $(\mathfrak{A}_{g,l})_{s_1}$ and A_0, then we take this weakly special subvariety as our new S. If again we find a positive-dimensional weakly special subvariety that is properly contained in S and contains some s_2 with the same property as s_1, we repeat this step. As the dimension has to drop in each step, this procedure will terminate after a finite number of steps and we have found the desired S (which is not necessarily unique).

Note that the endomorphism algebra of the generic fiber of $\mathcal{A} = \mathfrak{A}_{g,l} \times_{A_{g,l}} S \to S$ by construction contains a subalgebra that is isomorphic to the endomorphism algebra of A_0. However, it cannot be bigger since S contains some complex point s_n such that $\mathcal{A}_{s_n}$ admits a polarized isogeny to A_0 and specialization of endomorphisms is injective (see above). Hence, the endomorphism algebra of the generic fiber of $\mathcal{A} \to S$ is isomorphic to the endomorphism algebra of A_0.

This shows that S satisfies (i) and it satisfies (iii) by construction. For (ii), we remark that S contains a complex point s_n such that $\mathcal{A}_{s_n}$ admits a polarized isogeny to A_0 and that the Hecke orbit of s_n in $A_{g,l}$ consists only of points $s \in A_{g,l}(\mathbb{C})$ such that $(\mathfrak{A}_{g,l})_s$ admits a polarized isogeny to $\mathcal{A}_{s_n}$ (and hence to A_0). We can then apply Proposition 4.7 from [140]. $\square$

CHAPTER 6

The Zilber-Pink conjecture for complex abelian varieties

> Logic is a wonderful thing but
> doesn't always beat actual thought.
>
> T. Pratchett, *The Last Continent*

This chapter is joint work with Fabrizio Barroero.

6.1. Introduction

In this chapter, fields are always of characteristic 0 and curves and (sub)varieties will always be irreducible, but not necessarily geometrically irreducible. We work with the Zariski topology, therefore by open, dense, etc. we always mean Zariski open, Zariski dense, etc. except when we consider connected mixed Shimura (sub)varieties or (sub)data.

Let A be an abelian variety defined over an algebraically closed field K. A special subvariety of A is an irreducible component of an algebraic subgroup of A or, equivalently, a translate of an abelian subvariety by a torsion point. Arbitrary translates of abelian subvarieties are called cosets or weakly special subvarieties. Special subvarieties are also called torsion cosets.

The Manin-Mumford conjecture, proven by Raynaud [147], states that a subvariety of an abelian variety contains at most finitely many maximal special subvarieties. In particular, a non-special curve contains at most finitely many torsion points.

On the other hand, given a curve in an abelian variety, a dimension count suggests that it should not intersect a special subvariety of codimension at least 2. If one considers the union of all special subvarieties of codimension at least 2 and intersects it with a curve that is not contained in a proper special subvariety, one expects the intersection to be finite.

The pioneering work [22] of Bombieri, Masser and Zannier was one of the first to study this kind of problems and to pass from considering torsion points in subvarieties of algebraic groups to points lying in algebraic subgroups of appropriate codimension.

Indeed, Bombieri, Masser and Zannier proved that, given a curve defined over the algebraic numbers and contained in $\mathbb{G}_m^n$ but not in any of its proper (not necessarily torsion) cosets, it contains at most finitely many points that lie in an algebraic subgroup of $\mathbb{G}_m^n$ of codimension at least 2. The condition of not being contained in a proper coset was replaced by the necessary one of not lying in a proper torsion coset by Maurin [108] and independently by Bombieri, Habegger, Masser and Zannier in [21].

In the same paper [22], Bombieri, Masser and Zannier suggest that a possible analogue of their result for curves in $\mathbb{G}_m^n$ could hold for (families of) abelian varieties and that one could consider higher dimensional subvarieties and intersect them with algebraic subgroups of higher codimension.

A couple of years later, Zilber [195] independently stated a conjecture for semiabelian varieties of which the result of Bombieri, Masser and Zannier is a consequence. This is formulated in slightly different language and we are going to state it later. Similar conjectures for $\mathbb{G}_m^n$ were formulated by Bombieri, Masser and Zannier in [23].

We now consider an apparently weaker formulation of the same principle due to Pink. We introduce the following notation: For a non-negative integer k, we denote by $A^{[k]}$ the union of all special subvarieties of A of codimension at least k.

Pink conjectured in [141] that, if $V \cap A^{[\dim V+1]}$ is Zariski dense in V for a subvariety V of A, then V is contained in a proper special subvariety of A. The conjecture in its full generality is still open. If V is a curve and $K = \bar{\mathbb{Q}}$, it has been proven by Habegger and Pila in [70]. Previously, partial results have been obtained by Viada [184], [185], Rémond and Viada [160], Ratazzi [145], Carrizosa [28], [29] in combination with Rémond [155], [156], [157], and Galateau [48]. If V is a hypersurface, Pink's conjecture follows from the Manin-Mumford conjecture. If the dimension and codimension of V are at least 2, then all known results place additional restrictions on V or A, see for instance [31], [32], and [74].

In this chapter, we use a recent result of Gao in [54], which generalizes work by Rémond in [157], to reduce Zilber's conjecture to the case where everything is defined over $\bar{\mathbb{Q}}$. We even show that it can be reduced to Pink's formulation of the conjecture over $\bar{\mathbb{Q}}$. Furthermore, we prove the full conjecture in Corollary 6.1.7 if no abelian variety of dimension greater than 4 that is defined over $\bar{\mathbb{Q}}$ embeds into A. For example, the conjecture holds in a power of an elliptic curve with transcendental j-invariant. Combining Theorem 6.1.5 below with Theorem 1.1 in [70] yields the following theorem:

THEOREM 6.1.1. *Let A be an abelian variety defined over an algebraically closed field K (of characteristic 0) and let $V \subset A$ be a curve. Then $V \cap A^{[2]}$ is finite unless V is contained in a proper algebraic subgroup of A.*

As mentioned before, Pink's conjecture is implied by the following Conjecture 6.1.2 on unlikely or atypical intersections that was formulated by Zilber in [195] for semiabelian varieties. An overview of the topic of unlikely intersections is given in the book [193].

In order to state Conjecture 6.1.2, we introduce the notion of an atypical subvariety: Let A be an abelian variety defined over an algebraically closed field K and let V be a subvariety of A. A subvariety W of V is called atypical (for V in A) if W is an irreducible component of the intersection of V with a special subvariety of codimension at least $\dim V - \dim W + 1$. It is called maximal if it is not contained in any larger atypical subvariety.

CONJECTURE 6.1.2. *Let K be an algebraically closed field. Let A be an abelian variety defined over K and let V be a subvariety of A. Then V contains at most finitely many maximal atypical subvarieties.*

If V is a curve, then Conjecture 6.1.2 and Pink's conjecture are obviously equivalent.

It turns out that another equivalent formulation of Conjecture 6.1.2 is more suited to our proof strategy. In order to state it, we have to introduce the notions of defect and optimality of a subvariety.

DEFINITION 6.1.3. *If V is a subvariety of A, then there is a smallest special subvariety $\langle V\rangle$ containing V. We define the defect $\delta(V)$ of V to be* $\dim\langle V\rangle - \dim V$. *A subvariety W of V is called optimal for V in A if $\delta(U) > \delta(W)$ for every subvariety U with $W \subsetneq U \subset V$.*

Pink introduced the notion of defect in [141], while the concept of optimality was introduced in [70] by Habegger and Pila. The latter is motivated by Poizat's notion of *cd*-maximality in [143]. *cd*-maximality is the toric analogue of the notion of geodesic optimality, which we will introduce later. Using the concept of optimality, Habegger and Pila formulated the following conjecture, which is equivalent to Conjecture 6.1.2 by Lemma 2.7 in [70].

CONJECTURE 6.1.4. *Let K be an algebraically closed field and let d be a non-negative integer. Let A be an abelian variety defined over K and let V be a subvariety of A. Then V contains at most finitely many optimal subvarieties of defect at most d.*

In the statement of our results, we use the trace of an abelian variety with respect to a field extension of algebraically closed fields. This can be thought of as the largest abelian subvariety defined over the smaller field. See Definition 6.2.3 for a formal definition.

The following is the main result of this chapter:

THEOREM 6.1.5. *Let K be an algebraically closed field, let m be a non-negative integer and A an abelian variety defined over K with $K/\bar{\mathbb{Q}}$-trace (T, Tr). Then, if Conjecture 6.1.4 holds for some non-negative integer d and subvarieties of dimension at most m in T (over the field $\bar{\mathbb{Q}}$), it holds for the same d and subvarieties of dimension at most m in A (over K).*

Note that Habegger and Pila have shown in Corollary 9.10 in [70] that Conjecture 6.1.4 can be further reduced to the existence of sufficiently strong lower bounds for the size of the Galois orbits of optimal singletons over a field of definition that is finitely generated over $\mathbb{Q}$.

An analogue of Theorem 6.1.5 for powers of the multiplicative group was proven in [24] by Bombieri, Masser and Zannier. Note that in this case the ambient algebraic group is always defined over $\bar{\mathbb{Q}}$. In our situation, this corresponds to the special case where A is isomorphic to the base change of an abelian variety over $\bar{\mathbb{Q}}$.

Following a suggestion of Habegger, we prove Conjecture 6.1.4 in Theorem 6.7.1 if $K = \bar{\mathbb{Q}}$ and $d = 1$. This together with the preceding Theorem 6.1.5 and Theorem 1.1 in [70] implies the following corollary:

COROLLARY 6.1.6. *Let K be an algebraically closed field, let m be a non-negative integer and A an abelian variety defined over K. Then, Conjecture 6.1.4 holds for subvarieties of dimension at most m in A if either $m \leq 1$ or $d \leq 1$.*

Note that Conjecture 6.1.4 trivially holds for subvarieties of A of dimension or codimension 0. In codimension 1, every proper optimal subvariety is special, so Conjecture 6.1.4 follows from the theorem of Raynaud in [147] (Manin-Mumford conjecture). By Corollary 6.1.6, Conjecture 6.1.4 also holds for subvarieties of codimension 2 since every proper optimal subvariety of a subvariety of codimension 2 has defect at most 1; for the toric analogue see [23] (there proven over $\bar{\mathbb{Q}}$, then extended to $\mathbb{C}$ in [24]). Conjecture 6.1.4 has previously been proven for $K = \bar{\mathbb{Q}}$ and subvarieties of codimension 2 in powers of elliptic curves with complex multiplication (CM) (in [31]) and without CM (in [74]) as well as in arbitrary products of elliptic curves with CM (in [32]).

In particular, Conjecture 6.1.4 holds for abelian varieties of dimension at most 4 and by applying Theorem 6.1.5 we obtain the following corollary:

COROLLARY 6.1.7. *Let K be an algebraically closed field and A an abelian variety defined over K with $K/\bar{\mathbb{Q}}$-trace (T, Tr). If $\dim T \leq 4$, then Conjecture 6.1.4 holds for A.*

In the proof of Theorem 6.1.5, we use a double induction firstly on the dimension of A and secondly on the transcendence degree of its field of definition. If the transcendence degree of the field of definition is minimal in the sense that A is obtained as a geometric fiber of a certain universal family $\mathfrak{A}_{g,l} \to A_{g,l}$ of abelian varieties, then we apply Gao's result to reduce to abelian varieties of smaller dimension in Proposition 6.3.2.

We then use Rémond's results to increase the transcendence degree of the field of definition in Proposition 6.4.1. This part of the proof at some points resembles the proof of the main result in [24], albeit formulated rather differently.

The proofs of both propositions begin with the use of the fact that optimal subvarieties are geodesic-optimal, i.e., optimal with respect to the geodesic defect, which is the analogue of the defect if one replaces special by weakly special subvarieties. For abelian varieties, this has been proven by Habegger and Pila. It turns out that their proof can be adapted to show that the same holds if one considers a slightly different definition of the geodesic defect, where one replaces weakly special subvarieties by translates of abelian subvarieties by a torsion point plus a $\bar{\mathbb{Q}}$-point of the trace. We call it the $\bar{\mathbb{Q}}$-geodesic defect and $\bar{\mathbb{Q}}$-geodesic-optimal subvarieties are then the analogue of geodesic-optimal subvarieties for this defect.

To any ($\bar{\mathbb{Q}}$-)geodesic-optimal subvariety, there is an associated abelian subvariety. Thanks to the results of Gao and Rémond, this abelian subvariety lies in a finite set. If its dimension is positive, we can quotient out by it and use the inductive hypothesis. Otherwise, we either use the full strength of Gao's result to reduce to the trace or we use the inductive hypothesis on the transcendence degree of the field of definition concluding the proof.

As mentioned above, we also show that Conjecture 6.1.4 over an arbitrary algebraically closed field K can be reduced to Pink's formulation of the conjecture over $\bar{\mathbb{Q}}$. For the precise statement of our result, we now give a more articulated version of Pink's conjecture. Recall that we denote by $A^{[k]}$ the union of all special subvarieties of an abelian variety A of codimension at least k.

CONJECTURE 6.1.8. *Let K be an algebraically closed field and let d be a non-negative integer. Let A be an abelian variety defined over K and let V be a subvariety* *of A. If $V \cap A^{[\max\{\dim V+1, \dim A-d\}]}$ is Zariski dense in V, then V is contained in a proper special subvariety of A.*

The following statement draws a link between the above conjecture and Conjecture 6.1.4:

THEOREM 6.1.9. *Let K be an algebraically closed field, let m be a non-negative integer and A an abelian variety defined over K. Then, if Conjecture 6.1.8 holds for some non-negative integer d and subvarieties of dimension at most m in every abelian subvariety B of A, Conjecture 6.1.4 holds for the same d and subvarieties of dimension at most m in A.*

We prove Theorem 6.1.9 in Section 6.6. The proof is a direct application of Theorem 9.8(i) in [70]. As pointed out by Ullmo and Zannier, the analogous reduction can be done in the toric case by imposing additional multiplicative relations on the positive-dimensional atypical intersections. Combining Theorem 6.1.9 and Theorem 6.1.5 yields the following corollary:

COROLLARY 6.1.10. *Let K be an algebraically closed field, let m be a non-negative integer and A an abelian variety defined over K with $K/\bar{\mathbb{Q}}$-trace (T, Tr). Then, if Conjecture 6.1.8 holds for some non-negative integer d and subvarieties of dimension at most m in every abelian subvariety T' of T (over the field $\bar{\mathbb{Q}}$), Conjecture 6.1.4 holds for the same d and subvarieties of dimension at most m in A (over K).*

6.2. Preliminaries

In this section, we collect some results that are going to be useful in the proof of Theorem 6.1.5.

6.2.1. Definitions and a useful lemma. We are going to perform several base changes. We use the following notation:

DEFINITION 6.2.1. *Let V be a variety over a field K and let $K \subset L$ be a field extension. Then $V_L = V \times_K L$ is called the base change of V to L. We use analogous notation for the base change of morphisms between varieties.*

In the proof of Proposition 6.4.1, we argue by induction on the transcendence degree of our field of definition. Here is a basic fact used in the inductive step.

LEMMA 6.2.2. *Let $K \subset L$ be an extension of algebraically closed fields such that L has transcendence degree 1 over K. Let V be a variety over K and let W be a subvariety of V_L. Then there exists a subvariety W' of V with $\dim W' \leq \dim W + 1$ such that $W \subset W'_L$.*

PROOF. We can replace L by a finitely generated subextension of K of transcendence degree 1 over which W is defined. We can then find a curve C over K such that $K(C) = L$. We can also find a subvariety $\mathcal{W} \subset V \times_K C$ of dimension $\dim W + 1$ such that the generic fiber of $\mathcal{W}$ over C is W. The closure of the projection of $\mathcal{W}$ onto V is our W'. $\square$

We now give a formal definition of the trace of an abelian variety with respect to an extension of algebraically closed fields.

DEFINITION 6.2.3. ([33], Theorem 6.2) *Let $K \subset L$ be an extension of algebraically closed fields. Let A be an abelian variety defined over L. The L/K-trace of A is a pair (T, Tr) of an abelian variety T that is defined over K and a homomorphism of algebraic groups $\mathrm{Tr} : T_L \to A$ that is characterized uniquely by the fact that for every abelian variety T' that is defined over K and every homomorphism of algebraic groups $\sigma : T'_L \to A$, there is a homomorphism of algebraic groups $\tau : T' \to T$ such that $\sigma = \mathrm{Tr} \circ \tau_L$. The map Tr is a closed embedding.*

We introduce a notation that is going to make the exposition more agile.

DEFINITION 6.2.4. *Let K be an algebraically closed field and let A be an abelian variety over K. Let m and d be non-negative integers. We say that $\mathrm{ZP}(A, m, d)$ holds if Conjecture 6.1.4 holds for d and for all subvarieties V of A with $\dim V \le m$.*

We can then rephrase Theorem 6.1.5 as the implication

$$\mathrm{ZP}(T, m, d) \implies \mathrm{ZP}(A, m, d)$$

for non-negative integers m and d and an abelian variety A over an algebraically closed field K with $K/\bar{\mathbb{Q}}$-trace (T, Tr).

6.2.2. Smoothness and optimality under homomorphisms. We are going to project on quotients of abelian varieties by abelian subvarieties several times. The following two lemmata are going to be useful in this respect:

LEMMA 6.2.5. *Let K be an algebraically closed field and let $f : V \to W$ be a dominant morphism of algebraic varieties, defined over K. Then there exists $V_0 \subset V$ open and dense such that $f(V_0)$ is open and dense in W and $f|_{V_0} : V_0 \to f(V_0)$ is smooth.*

PROOF. By Corollary II.8.16 in [71], we can find $V_1 \subset V$ open, dense, and non-singular. By Theorem 10.19 in [58], $f(V_1)$ contains V_2 open and dense in $\overline{f(V_1)} = \overline{f(V)} = W$. By generic smoothness (Corollary III.10.7 in [71]), we can find $V_3 \subset V_2$ open and dense in V_2 and hence in W such that $V_0 = f|_{V_1}^{-1}(V_3)$ is open and dense in V and $f|_{V_0} : V_0 \to V_3 = f(V_0)$ is smooth. □

LEMMA 6.2.6. *Let K be an algebraically closed field. Let $f : A \to A'$ be a homomorphism of algebraic groups between abelian varieties defined over K. Then the following hold:*

(1) Let V be a subvariety of A. Suppose that $V_0 \subset V$ is open and dense in V such that $f(V_0)$ is open and dense in $f(V)$ and $f|_{V_0} : V_0 \to f(V_0)$ is smooth. Let W be a subvariety of V that is optimal for V in A and intersects V_0. If $\langle W \rangle$ contains a translate of a component of $\ker f$, then $f(W)$ has defect at most $\delta(W)$ and is optimal for $f(V)$ in A'.

(2) If f has finite kernel and $\mathrm{ZP}(A', m, d)$ holds, then $\mathrm{ZP}(A, m, d)$ holds.

PROOF. For (1), let W be optimal for V in A such that $W \cap V_0 \neq \emptyset$, where $f|_{V_0} : V_0 \to f(V_0)$ is smooth of relative dimension n. Let U be a subvariety of A' such that $f(W) \subset U \subset f(V)$ and $\delta(U) \le \delta(f(W))$. Since $f(\langle W \rangle)$ contains $\langle f(W) \rangle$ and $\langle W \rangle$ contains a translate of a component of $\ker f$, we have

$$\delta(U) \le \delta(f(W)) \le \dim f(\langle W \rangle) - \dim f(W) = \dim \langle W \rangle - \dim \ker f - \dim f(W).$$

Let now U' be an irreducible component of $f^{-1}(U) \cap V = f|_V^{-1}(U)$ that contains W. We have $U' \cap V_0 \neq \emptyset$ since it contains $W \cap V_0$. Furthermore, $U' \cap V_0$ is an irreducible component of $f|_{V_0}^{-1}(U \cap f(V_0))$. Since $f|_{V_0} : V_0 \to f(V_0)$ is smooth of relative dimension n, it follows that $\dim U' = \dim(U' \cap V_0) = \dim(U \cap f(V_0)) + n = \dim U + n$. Taking $U = f(W)$ and using that $W \subset U'$ shows that $\dim W \leq \dim f(W) + n$. If U is again an arbitrary subvariety as above, we deduce that

$$\delta(U) \leq \delta(f(W)) \leq \dim\langle W\rangle - \dim\ker f - \dim W + n. \tag{6.2.1}$$

After noting that $n \leq \dim\ker f$, we deduce that the defect of $f(W)$ is at most the defect of W.

Assume now that $f(W)$ is not optimal for $f(V)$ in A'. It follows that we can choose U as above with $f(W) \subsetneq U$. Since $\dim U > \dim f(W)$, we have $W \subsetneq U'$ and so $\delta(U') > \delta(W)$ by the optimality of W. Moreover, we certainly have $\dim\langle U'\rangle \leq \dim\langle U\rangle + \dim\ker f$, so it follows that $\delta(U') \leq \delta(U) + \dim\ker f - n$. Using (6.2.1), we deduce that

$$\delta(U) \leq \delta(W) - \dim\ker f + n < \delta(U') - \dim\ker f + n \leq \delta(U),$$

a contradiction.

We can now prove (2) by induction on m. For the base step of the induction, note that $\mathrm{ZP}(A, 0, d)$ holds trivially for all d. Let now V be a subvariety of A of dimension m and let W be an optimal subvariety for V in A such that W has defect at most d. By Lemma 6.2.5, we can find $V_0 \subset V$ open and dense such that $f(V_0)$ is open and dense in $f(V)$ and $f|_{V_0} : V_0 \to f(V_0)$ is smooth.

If $W \subset V \backslash V_0$, then W is contained in one of the finitely many components of $V \backslash V_0$. The dimension of that component is at most $m-1$ and W has defect at most d and is optimal for that component in A, so we are done by induction. Otherwise, we have $W \cap V_0 \neq \emptyset$. Since $\ker f$ is finite, a translate of one of its components is trivially contained in $\langle W\rangle$, so we can apply (1) to find that $f(W)$ is optimal for $f(V)$ in A' and has defect at most d. As f has finite kernel, we have $\dim f(V) = \dim V$, so it follows from $\mathrm{ZP}(A', m, d)$ that $f(W)$ belongs to a finite set of varieties. The same follows for W since W is an irreducible component of $f^{-1}(f(W))$ because of the equality $\dim W = \dim f(W) = \dim f^{-1}(f(W))$. □

6.2.3. Abelian schemes. We start this subsection by associating to any abelian variety over an algebraically closed field of characteristic 0 a "subfamily" of the universal family of principally polarized abelian varieties of fixed dimension with some fixed level structure.

We denote by $A_{g,l}$ the moduli space of principally polarized abelian varieties of dimension g with symplectic level l-structure over $\bar{\mathbb{Q}}$. For $l \geq 3$, this is a fine moduli space and we denote by $\mathfrak{A}_{g,l}$ the corresponding universal family over $A_{g,l}$.

LEMMA 6.2.7. *Let K be an algebraically closed field. Let A be an abelian variety of dimension g defined over K. Fix a natural number $l \geq 3$. Then there exists a subvariety $S \subset A_{g,l}$ with the following property: Let $\mathcal{A} = \mathfrak{A}_{g,l} \times_{A_{g,l}} S$ and let A' be the generic fiber of $\mathcal{A} \to S$. There exists a field embedding $\bar{\mathbb{Q}}(S) \hookrightarrow K$ such that A is isogenous to A'_K.*

PROOF. Over K, the abelian variety A is isogenous to a principally polarized abelian variety by Corollary 1 on p. 234 of [118] and so we can assume without loss of generality that A is itself principally polarized. We can find a field $K_0 \subset K$ and an abelian variety B defined over K_0 such that $\bar{\mathbb{Q}} \subset K_0$, K_0 is finitely generated over $\bar{\mathbb{Q}}$, and $A = B_K$. Without loss of generality, we can assume that all torsion points of B of order l are K_0-rational. We can find a normal variety V, defined over $\bar{\mathbb{Q}}$, with $\bar{\mathbb{Q}}(V) = K_0$. By spreading out (see Theorem 3.2.1 and Table 1 on pp. 306–307 of [144]), we find an abelian scheme $\mathcal{B} \to V$ with generic fiber B (after maybe replacing V by an open dense subset). The principal polarization of B gives a principal polarization of $\mathcal{B} \to V$ by the argument on p. 6 of [45]. Among the l-torsion points of B, we choose a symplectic basis with respect to the Weil pairing induced by the principal polarization. The elements of this basis extend to l-torsion sections of $\mathcal{B} \to V$. In this way, $\mathcal{B} \to V$ becomes a principally polarized abelian scheme with symplectic level l-structure as defined in the Appendix to Chapter 7 of [120].

Since $A_{g,l}$ is a fine moduli space by the Appendix to Chapter 7 of [120], we get a morphism $\iota : V \to A_{g,l}$, defined over $\bar{\mathbb{Q}}$, such that $\mathcal{B}$ is isomorphic to $\mathfrak{A}_{g,l} \times_{A_{g,l}} V$. Let S be the closure of $\iota(V)$, let $\mathcal{A} = \mathfrak{A}_{g,l} \times_{A_{g,l}} S$ and let A' be the generic fiber of $\mathcal{A} \to S$. It follows from the universal property of the fiber product that $\mathcal{B}$ is isomorphic to $\mathcal{A} \times_S V$. The dominant morphism $\iota : V \to S$ yields a field embedding $\bar{\mathbb{Q}}(S) \hookrightarrow K_0$. Passing to the generic fiber over V shows that B is isomorphic to A'_{K_0}. □

Let S be a variety over an algebraically closed field K. Given an abelian scheme $\mathcal{A} \to S$ with geometric generic fiber A, we need to extend abelian subvarieties and torsion points of A and K-points of the trace of A to abelian subschemes, torsion sections, and constant sections, possibly after a base change. Recall that an irreducible subgroup scheme $\mathcal{B}$ of $\mathcal{A}$ is called an abelian subscheme if $\mathcal{B} \to S$ is flat, proper, and dominant.

Lemma 6.2.8. *Let S be a normal variety and $\mathcal{A} \to S$ an abelian scheme, both defined over an algebraically closed field K. Let ξ be the generic point of S. Suppose that every l-torsion point of $(\mathcal{A}_\xi)_{\overline{K(S)}}$ is $K(S)$-rational for some natural number $l \geq 3$, where $\overline{K(S)}$ denotes a fixed algebraic closure of $K(S)$. Then every abelian subvariety B of $(\mathcal{A}_\xi)_{\overline{K(S)}}$ is the geometric generic fiber of an abelian subscheme $\mathcal{B} \subset \mathcal{A}$.*

Proof. Fix an abelian subvariety B of $(\mathcal{A}_\xi)_{\overline{K(S)}}$. It is a consequence of the Poincaré reducibility theorem that there exists an endomorphism ψ of $(\mathcal{A}_\xi)_{\overline{K(S)}}$ such that B is the irreducible component of $\ker(\psi)$ containing the neutral element. By Theorem 2.4 in [170], every endomorphism of $(\mathcal{A}_\xi)_{\overline{K(S)}}$ is the base change of an endomorphism of $\mathcal{A}_\xi$. It follows that every abelian subvariety of $(\mathcal{A}_\xi)_{\overline{K(S)}}$ is the base change of an abelian subvariety of $\mathcal{A}_\xi$. We identify ψ and B with the corresponding endomorphism and abelian subvariety of $\mathcal{A}_\xi$ respectively.

By Theorem 3.2.1(iii) in [144] (cf. Théorème 8.8.2(i) in [65]), the endomorphism ψ spreads out to a morphism from $\mathcal{A}_U = \mathcal{A} \times_S U$ to itself for $U \subset S$ open and dense. This morphism has to be an endomorphism by Corollary 6.4 on p. 117 of [120]. By Proposition 2.7 in Chapter I of [45], as S is normal, this endomorphism extends to an endomorphism $\Psi : \mathcal{A} \to \mathcal{A}$.

We get a closed subgroup scheme $\ker(\Psi)$ of $\mathcal{A}$. Let $\ker(\Psi)^0$ be the functor defined in Section 3 of Exposé VIB in [6]. We want to apply Corollaire 4.4 from Exposé VIB in [6] by verifying condition (ii).

Fix $s \in S$. The fiber $\ker(\Psi)_s$ is an algebraic group over a field (of characteristic 0 as always) and hence smooth. Since Ψ is proper, $\Psi(\mathcal{A})$ is a closed irreducible subscheme of $\mathcal{A}$ and it follows from the Fiber Dimension Theorem (Lemma 14.109 in [58]) applied to the morphism $\Psi(\mathcal{A}) \to S$ that $\dim \Psi(\mathcal{A}_s) \geq \dim \Psi(\mathcal{A}_\xi)$. Similarly, we have $\dim \ker(\Psi)_s \geq \dim B$. Furthermore, we have $\dim \mathcal{A}_s = \dim \ker(\Psi)_s + \dim \Psi(\mathcal{A}_s)$. But then it follows from $\dim \Psi(\mathcal{A}_\xi) + \dim B = \dim \mathcal{A}_\xi = \dim \mathcal{A}_s$ that $\dim \ker(\Psi)_s = \dim B$. This means that the function $s \mapsto \dim \ker(\Psi)_s$ is constant on S.

Therefore, we can apply Corollaire 4.4 of Exposé VIB in [6] to find that $\ker(\Psi)^0$ is represented by an open subgroup scheme of $\ker(\Psi)$, which we also denote by $\ker(\Psi)^0$. By the same Corollaire, $\ker(\Psi)^0$ is smooth over S. By definition, the generic fiber of $\ker(\Psi)^0$ is equal to B.

The number of geometrically irreducible components of the fibers of $\ker(\Psi)$ is uniformly bounded. Therefore, for some large N, we have that $\ker(\Psi)^0$ is equal to the image of $\ker(\Psi)$ under the multiplication-by-N morphism. As this morphism is proper, it follows that $\ker(\Psi)^0$ is closed in $\mathcal{A}$ and therefore the morphism $\ker(\Psi)^0 \to S$ is proper. Since $\ker(\Psi)^0$ is smooth over S, it is flat over S. As its generic fiber is irreducible and it is flat over S, $\ker(\Psi)^0$ is irreducible as well by Proposition 2.3.4(iii) in [64] and Section 2.1.8 of Chapter 0 of [61]. Since $\ker(\Psi)^0$ is a subgroup scheme that dominates S, this shows that $\ker(\Psi)^0$ is an abelian subscheme of $\mathcal{A}$ with generic fiber equal to B as desired. □

Lemma 6.2.8 does not hold if the base variety S and the abelian scheme $\mathcal{A}$ are allowed to be arbitrary. We provide a non-trivial counterexample: We let $S \subset \mathbb{A}^1_K \backslash \{0,1\} \times_K \mathbb{A}^1_K$ be defined by the equation $(\lambda+1)^2\lambda = \mu^2$ in the affine coordinates (λ, μ) on $\mathbb{A}^2_K$. Let ξ be the generic point of S. We consider two elliptic schemes $\mathcal{E}$ and $\mathcal{E}'$ over S that are defined in $\mathbb{P}^2_K \times_K S$ by equations $y^2z = x(x-z)(x-\lambda z)$ and $\lambda y'^2 z' = x'(x'-z')(x'-\lambda z')$ in the projective coordinates $[x : y : z]$ and $[x' : y' : z']$ respectively. We set $\mathcal{A} = \mathcal{E} \times_S \mathcal{E}'$ and let B be the abelian subvariety of $\mathcal{A}_\xi \subset \mathbb{P}^2_{K(S)} \times_{K(S)} \mathbb{P}^2_{K(S)}$ defined by the equations $xz' = x'z$ and $yz' = \frac{\mu}{\lambda+1} y'z$. If $p = (-1, 0) \in S(K) \subset \mathbb{A}^2_K(K) = K^2$ is the singular point of S, then B extends to an abelian subscheme of $\mathcal{A} \times_S (S \backslash \{p\})$, but the fiber over p of the closure of B in $\mathcal{A}$ has two irreducible components.

Lemma 6.2.9. *Let S be a normal variety and $\mathcal{A} \to S$ an abelian scheme, both defined over an algebraically closed field K. Let ξ be the generic point of S. Suppose that every l-torsion point of $(\mathcal{A}_\xi)_{\overline{K(S)}}$ is $K(S)$-rational for some natural number $l \geq 3$, where $\overline{K(S)}$ denotes a fixed algebraic closure of $K(S)$. Let N be a fixed natural number.*

Then there exists a normal variety S' over K with generic point η and a finite surjective étale morphism $S' \to S$ such that the following hold for $\mathcal{A}' = \mathcal{A} \times_S S'$ (after fixing an algebraic closure $\overline{K(S')}$ of $K(S')$):

(1) Every torsion point of $(\mathcal{A}'_\eta)_{\overline{K(S')}}$ of order N is the geometric generic fiber of a torsion section $S' \to \mathcal{A}'$.

(2) *Let* (T, Tr) *denote the* $\overline{K(S')}/K$*-trace of* $(\mathcal{A}'_\eta)_{\overline{K(S')}}$. *Then* $\mathrm{Tr}\left(T_{\overline{K(S')}}\right)$ *is the geometric generic fiber of an abelian subscheme* $\mathcal{T}$ *of* $\mathcal{A}'$ *that is isomorphic (as an abelian scheme) to* $T \times_K S'$.

Proof. Let $\epsilon : S \to \mathcal{A}$ be the zero section, let g denote the relative dimension of $\mathcal{A} \to S$ and let $[N] : \mathcal{A} \to \mathcal{A}$ denote the multiplication-by-N morphism. The morphism $[N]^{-1}(\epsilon(S)) \to S$ is finite and étale since ϵ is a closed embedding and $[N]$ is finite and étale by Proposition 20.7 in [111].

Using Proposition 2.3.4(iii) in [64], we deduce that every irreducible component of $[N]^{-1}(\epsilon(S))$ surjects onto S. Furthermore, $[N]^{-1}(\epsilon(S))$ is normal by Proposition 11.3.13(ii) in [65] since it is étale over the normal variety S. Hence, no two distinct irreducible components of $[N]^{-1}(\epsilon(S))$ can intersect each other. Therefore, every irreducible component of $[N]^{-1}(\epsilon(S))$ is finite and étale over S.

We can then achieve (1) by successively base changing to irreducible components of $[N]^{-1}(\epsilon(S))$ which have degree > 1 over S and using at the end that a finite étale morphism of degree 1 is an isomorphism. We take S' to be the base variety we end up with.

For (2), we then first use Lemma 6.2.8 to identify $\mathrm{Tr}\left(T_{\overline{K(S')}}\right)$ with the geometric generic fiber of an abelian subscheme $\mathcal{T}$ of $\mathcal{A}'$. Second, we note that Tr is the base change of a homomorphism $T_{K(S')} \to \mathcal{A}'_\eta$ by Theorem 2.4 in [170]. We denote this homomorphism also by Tr. Third, arguing as in the proof of Lemma 6.2.8, we can extend the induced isomorphism of abelian varieties $T_{K(S')} \to \mathrm{Tr}\left(T_{K(S')}\right)$ to an isomorphism of abelian schemes $T \times_K S' \to \mathcal{T}$. □

6.2.4. Defect and optimality. Here we show that extending the field of definition cannot give new optimal subvarieties.

Lemma 6.2.10. *Let* $K \subset L$ *be an extension of algebraically closed fields. Let* A *be an abelian variety defined over* K *and let* V *be a subvariety of* A. *If* W *is an optimal subvariety for* V_L *in* A_L, *then there exists an optimal subvariety* W' *for* V *in* A *such that* $W = (W')_L$ *and* $\delta(W) = \delta(W')$.

Proof. Since A is defined over K, which is algebraically closed, we have that any special subvariety of A_L is the base change of a special subvariety of A. Therefore, if V is a subvariety of A, any optimal subvariety for V_L in A_L is an irreducible component of an intersection $V_L \cap H_L$ for some special subvariety H of A and is then the base change of a subvariety $W \subset V$ that must be optimal for V in A and of the same defect. □

We also introduce a new kind of defect.

Definition 6.2.11. *Let* $K \subset L$ *be an extension of algebraically closed fields. Let* A *be an abelian variety defined over* L *with* L/K*-trace* (T, Tr). *For a subvariety* V *of* A *we define* $\langle V\rangle_{K,geo}$ *to be the smallest translate of an abelian subvariety of* A *by a point in* $\mathrm{Tr}(T(K)) + A_{\mathrm{tors}}$ *that contains* V. *We call* K*-geodesic defect the difference* $\delta_{K,geo}(V) = \dim\langle V\rangle_{K,geo} - \dim V$. *If* $W \subset V \subset A$, *we say that* W *is* K*-geodesic-optimal for* V *in* A *if* $\delta_{K,geo}(U) > \delta_{K,geo}(W)$ *for every subvariety* U *with* $W \subsetneq U \subset V$.

In the case $K = L$, we drop K in the notation as we are considering usual weakly special subvarieties, the geodesic defect, and geodesic optimality as defined in [70].

Note that A is isogenous to $T_L \times B$ for an abelian variety B such that there exists no non-trivial homomorphism between T_L and B. Therefore the intersection of two translates of abelian subvarieties of A by points in $\mathrm{Tr}(T(K)) + A_{\mathrm{tors}}$ is again a finite union of translates of abelian subvarieties of A by points in $\mathrm{Tr}(T(K)) + A_{\mathrm{tors}}$, so $\langle V\rangle_{K,geo}$ is well defined.

In [70], Habegger and Pila defined the defect condition for subvarieties of complex abelian varieties. If W and V are subvarieties of A such that $W \subset V$, then the condition says that $\delta(V) - \delta_{geo}(V) \leq \delta(W) - \delta_{geo}(W)$. They then showed that the fact that this holds in this setting implies that optimal subvarieties are also geodesic-optimal. Here we do the same for $\delta_{K,geo}$ in place of δ_{geo} and show that optimal subvarieties are also K-geodesic-optimal.

LEMMA 6.2.12. *In the setting of Definition 6.2.11, let W and V be subvarieties of A such that $W \subset V$. Then, the following hold:*

(1) We have $\delta(V) - \delta_{K,geo}(V) \leq \delta(W) - \delta_{K,geo}(W)$.
(2) If $W \subset V$ is optimal for V in A, then it is K-geodesic-optimal for V in A.

PROOF. The deduction of (2) from (1) is purely formal and identical to the proof of Proposition 4.5 in [70]: Let U be a subvariety of A such that $W \subset U \subset V$, $\delta_{K,geo}(U) \leq \delta_{K,geo}(W)$, and U is K-geodesic-optimal for V in A. Then it follows from (1), applied to $W \subset U$, that $\delta(U) - \delta_{K,geo}(U) \leq \delta(W) - \delta_{K,geo}(W)$ and so

$$\delta(U) = \delta_{K,geo}(U) + \delta(U) - \delta_{K,geo}(U) \leq \delta_{K,geo}(W) + \delta(W) - \delta_{K,geo}(W) = \delta(W).$$

Since W is optimal for V in A, we deduce that $U = W$ and so W is K-geodesic-optimal for V in A.

It remains to prove (1): The defect condition in this case amounts to proving that

$$\dim\langle V\rangle - \dim\langle V\rangle_{K,geo} \leq \dim\langle W\rangle - \dim\langle W\rangle_{K,geo}.$$

For this, we can just copy (almost verbatim) the proof of Proposition 4.3(ii) in [70]: The coset $\langle V\rangle_{K,geo}$ is a translate of an abelian subvariety B of A. We write $\phi : A \to A/B$ for the quotient morphism. We fix an auxiliary point $w \in W(L)$.

We remark that $\langle\{w\}\rangle + B$ is a torsion coset that contains $w + B$. As $w \in V(L)$, we also have $V \subset w + B$ and thus $\langle V\rangle \subset \langle\{w\}\rangle + B$. We apply ϕ, which has kernel B, to find that $\phi(\langle V\rangle) \subset \phi(\langle\{w\}\rangle) \subset \phi(\langle W\rangle)$. Note that $\langle V\rangle$ contains a translate of B, namely $\langle V\rangle_{K,geo}$. We conclude that

$$\dim\langle V\rangle - \dim\langle V\rangle_{K,geo} = \dim\langle V\rangle - \dim B = \dim\phi(\langle V\rangle) \leq \dim\phi(\langle W\rangle). \tag{6.2.2}$$

The torsion coset $\langle W\rangle$ is the translate of an abelian subvariety C of A by a torsion point. The fibers of $\phi|_{\langle W\rangle} : \langle W\rangle \to \phi(\langle W\rangle)$ contain translates of $B \cap C$. We find that $\dim\phi(\langle W\rangle) \leq \dim\langle W\rangle - \dim B \cap C$. We observe that $\dim B \cap C \geq \dim\langle W\rangle_{K,geo}$ and so $\dim\phi(\langle W\rangle) \leq \dim\langle W\rangle - \dim\langle W\rangle_{K,geo}$. We now combine this bound with (6.2.2) to deduce that

$$\dim\langle V\rangle - \dim\langle V\rangle_{K,geo} \leq \dim\langle W\rangle - \dim\langle W\rangle_{K,geo}$$

as desired.

□

The geodesic defect of a subvariety of the connected mixed Shimura variety $(\mathfrak{A}_{g,l})_{\mathbb{C}}$, which we define below, is linked to the $\mathbb{C}$-geodesic defect of the components of its geometric generic fiber.

LEMMA 6.2.13. *Suppose that $\mathcal{U}$ is a subvariety of $(\mathfrak{A}_{g,l})_{\mathbb{C}}$ and S is its image under the projection to $(A_{g,l})_{\mathbb{C}}$. Let U be an irreducible component of the fiber of $\mathcal{U}$ over the geometric generic point of S. We define the geodesic defect $\delta_{geo}(\mathcal{U})$ of $\mathcal{U}$ to be $\dim\langle S\rangle_{geo} - \dim S + \dim\langle U\rangle_{\mathbb{C},geo} - \dim U$, where $\langle S\rangle_{geo}$ is the smallest weakly special subvariety of $(A_{g,l})_{\mathbb{C}}$ (see [179] for a definition and a characterization) that contains S. This definition of geodesic defect is independent of the choice of U and agrees with δ_{ws} as defined in Definition 8.1(i) in [54].*

PROOF. The independence from the choice of U follows from the fact that the irreducible components of the fiber of $\mathcal{U}$ over the geometric generic point of S form one orbit under the action of the Galois group $\mathrm{Gal}\left(\overline{\mathbb{C}(S)}/\mathbb{C}(S)\right)$ and $\langle\cdot\rangle_{\mathbb{C},geo}$ commutes with the action of $\mathrm{Gal}\left(\overline{\mathbb{C}(S)}/\mathbb{C}(S)\right)$.

In Definition 8.1(i) in [54], the defect $\delta_{ws}(\mathcal{U})$ is defined as $\dim \mathcal{U}^{biZar} - \dim \mathcal{U}$, where $\mathcal{U}^{biZar}$ is the smallest bi-algebraic subvariety of $(\mathfrak{A}_{g,l})_{\mathbb{C}}$ containing $\mathcal{U}$. By Proposition 5.3 in [53], this is equal to $\dim\langle S\rangle_{geo} - \dim S + \dim \mathcal{W} - \dim \mathcal{U}$, where $\mathcal{W}$ is the smallest generically special subvariety of sg type (as defined in Definition 1.5 in [53]) of $(\mathfrak{A}_{g,l})_{\mathbb{C}} \times_{(A_{g,l})_{\mathbb{C}}} S$ containing $\mathcal{U}$.

It is enough to show that $\dim\langle U\rangle_{\mathbb{C},geo} - \dim U = \dim \mathcal{W} - \dim \mathcal{U}$. By looking at the geometric generic fiber of $\mathcal{W}$, we see that $\dim\langle U\rangle_{\mathbb{C},geo} - \dim U \leq \dim \mathcal{W} - \dim \mathcal{U}$. For the inequality in the other direction to hold, we need to know that after a finite surjective base change we can extend abelian subvarieties and torsion points of the geometric generic fiber and $\mathbb{C}$-points of the trace to abelian subschemes, torsion sections, and constant sections respectively. This is ensured by Lemma 6.2.8 and Lemma 6.2.9; note that after a finite surjective base change we can assume the base to be normal. □

DEFINITION 6.2.14. *If $\mathcal{W} \subset \mathcal{V}$ are subvarieties of $(\mathfrak{A}_{g,l})_{\mathbb{C}}$, we say that $\mathcal{W}$ is geodesic-optimal for $\mathcal{V}$ in $(\mathfrak{A}_{g,l})_{\mathbb{C}}$ if $\delta_{geo}(\mathcal{U}) > \delta_{geo}(\mathcal{W})$ for every subvariety $\mathcal{U}$ with $\mathcal{W} \subsetneq \mathcal{U} \subset \mathcal{V}$.*

6.3. A statement in the universal family

The following result is a fundamental tool for our proof. It relies on a result of Gao, which is formulated in the language of mixed Shimura varieties. We show that, in the special setting of an abelian variety that is the geometric generic fiber of a "subfamily" of the universal family, Gao's result yields a strengthening of what is sometimes called a "Structure Theorem" proven by Rémond for abelian varieties in [157]. The analogous statement for powers of the multiplicative group was proven by Poizat in Corollaire 3.7 in [143] and independently by Bombieri, Masser and Zannier in [23]. Recall that $A_{g,l}$ is a variety over $\bar{\mathbb{Q}}$.

THEOREM 6.3.1. *Let S be a subvariety of $A_{g,l}$ and $\mathcal{A} = \mathfrak{A}_{g,l} \times_{A_{g,l}} S$. Let A be the geometric generic fiber of $\mathcal{A}$ and V a subvariety of A and let (T, Tr) be the $K/\bar{\mathbb{Q}}$-trace of A, where K is a fixed algebraic closure of $\bar{\mathbb{Q}}(S)$. There is a finite set* *of pairs* (q_0, H)*, where* $q_0 \in A(K)$ *is a torsion point and* H *is an abelian subvariety of* A*, and a finite union* Z *of proper subvarieties of* V *such that for every optimal* $W \subset V$ *one of the following holds:*

(1) W *is contained in* Z*, or*
(2) *there exists some point* $t \in \mathrm{Tr}(T(\bar{\mathbb{Q}}))$ *such that* W *is an irreducible component of* $(t + q_0 + H) \cap V$ *and* $t + q_0 + H \subset \langle W \rangle$*.*

Proof. We want to apply a result of Gao (Theorem 8.2 in [54]). For this, we need to make a base change. Let K' be a fixed algebraic closure of the function field $\mathbb{C}(S_{\mathbb{C}})$ of $S_{\mathbb{C}}$. Both K and $\mathbb{C}$ embed into K' and the intersection of their images is $\bar{\mathbb{Q}}$ since the transcendence degree of $K'/\mathbb{C}$ is equal to the transcendence degree of $K/\bar{\mathbb{Q}}$.

Let W be optimal for V in A. We want to show that $W_{K'}$ is optimal for $V_{K'}$ in $A_{K'}$. If U is an optimal subvariety for $V_{K'}$ containing $W_{K'}$ and satisfying $\delta(U) \leq \delta(W_{K'})$, then, by Lemma 6.2.10, U is the base change of a subvariety of V of the same defect that is optimal for V in A. Because of the optimality of W for V in A, that subvariety has to be equal to W, so $U = W_{K'}$.

By Lemma 6.2.12, $W_{K'}$ is also $\mathbb{C}$-geodesic-optimal for $V_{K'}$ in $A_{K'}$.

Suppose first that $W_{K'} \subset \sigma(V_{K'})$ for some $\sigma \in \mathrm{Gal}(K'/\mathbb{C}(S_{\mathbb{C}}))$ with $\sigma(V_{K'}) \neq V_{K'}$. Then $W_{K'}$ is contained in $V_{K'} \cap \sigma(V_{K'}) \subsetneq V_{K'}$. Let $Z_\sigma \subset A$ be maximal among all finite unions of subvarieties $Z' \subset A$ with $Z'_{K'} \subset V_{K'} \cap \sigma(V_{K'})$ (and equal to the closure of the union of all such Z'). Then Z_σ is a finite union of proper subvarieties of V and contains W. We set $Z = \cup_{\sigma(V_{K'}) \neq V_{K'}} Z_\sigma$; the union is finite since $\sigma(V_{K'})$ varies in a finite set. We deduce that $W \subset Z$, so (1) is satisfied.

From now on, we assume that $W_{K'} \subset \sigma(V_{K'})$ only holds if $\sigma(V_{K'}) = V_{K'}$ (for $\sigma \in \mathrm{Gal}(K'/\mathbb{C}(S_{\mathbb{C}}))$), and we want to prove that (2) holds.

The subvarieties $W_{K'}$ and $V_{K'}$ are irreducible components of the base change of subvarieties of the generic fiber of $\mathcal{A}_{\mathbb{C}}$. We define $\mathcal{W}$ and $\mathcal{V}$ to be the closures of these two subvarieties in $\mathcal{A}_{\mathbb{C}}$. Note that they are subvarieties of dimension $\dim S + \dim W$ and $\dim S + \dim V$ respectively and that they dominate $S_{\mathbb{C}}$.

In Lemma 6.2.13, we defined the geodesic defect of subvarieties of $(\mathfrak{A}_{g,l})_{\mathbb{C}}$ and we have seen that it coincides with δ_{ws} of [54]. Let $\mathcal{U} \subset \mathcal{V}$ be a geodesic-optimal subvariety for $\mathcal{V}$ that contains $\mathcal{W}$ and satisfies $\delta_{geo}(\mathcal{U}) \leq \delta_{geo}(\mathcal{W})$. Using that $\mathcal{W} \subset \mathcal{U}$, we find that there exists an irreducible component U of the geometric generic fiber of $\mathcal{U}$ that contains $W_{K'}$ and satisfies

$$\begin{aligned}\delta_{\mathbb{C},geo}(U) &= \delta_{geo}(\mathcal{U}) - \dim\langle S\rangle_{geo} + \dim S \\ &\leq \delta_{geo}(\mathcal{W}) - \dim\langle S\rangle_{geo} + \dim S = \delta_{\mathbb{C},geo}(W_{K'}).\end{aligned}$$

Since $\mathcal{U} \subset \mathcal{V}$, we can deduce that $U \subset \sigma(V_{K'})$ for some $\sigma \in \mathrm{Gal}(K'/\mathbb{C}(S_{\mathbb{C}}))$. Since $W_{K'} \subset U \subset \sigma(V_{K'})$, we must have $\sigma(V_{K'}) = V_{K'}$ by our assumption from above.

Since $W_{K'}$ is $\mathbb{C}$-geodesic-optimal for $V_{K'}$ in $A_{K'}$, it follows that $W_{K'} = U$ and therefore $\mathcal{W} = \mathcal{U}$. Hence, $\mathcal{W}$ is geodesic-optimal for $\mathcal{V}$ in the connected mixed Shimura variety $(\mathfrak{A}_{g,l})_{\mathbb{C}}$.

Let $\mathcal{W}^{biZar}$ be the smallest bi-algebraic subvariety of $(\mathfrak{A}_{g,l})_{\mathbb{C}}$ that contains $\mathcal{W}$. It is determined by a tuple $(Q, \mathcal{Y}^+, N, \tilde{y})$, where $(Q, \mathcal{Y}^+)$ is a connected mixed Shimura subdatum of $(\mathrm{GSp}_{2g} \ltimes \mathbb{Q}^{2g}, \mathbb{H}_g \times \mathbb{C}^g)$, N is a normal subgroup of the derived subgroup Q^{der}, and $\tilde{y} \in \mathcal{Y}^+$. Here $\mathbb{H}_g$ denotes the Siegel upper half space. Thanks to Theorem

8.2 in [54], we know that the triple $(Q, \mathcal{Y}^+, N)$ lies in a finite set that does not depend on $\mathcal{W}$. By Proposition 5.3 in [53], $\mathcal{W}^{biZar}$ is a generically special subvariety of sg type (as defined in Definition 1.5 in [53]) of $(\mathfrak{A}_{g,l})_\mathbb{C} \times_{(A_{g,l})_\mathbb{C}} \pi(\mathcal{W}^{biZar})$, where $\pi : (\mathfrak{A}_{g,l})_\mathbb{C} \to (A_{g,l})_\mathbb{C}$ is the structural morphism, so up to finite surjective base change a translate of an abelian subscheme by a torsion section and a constant section.

Looking at the proof of Proposition 3.3 on p. 240 of [51], we see that the abelian subscheme and the torsion section are uniquely determined by $(Q, \mathcal{Y}^+, N)$. Note that $\tilde{y}_G$ in the proof of Proposition 3.3 in [51] can be assumed fixed as $\pi(\mathcal{W}) = S_\mathbb{C}$ is independent of $\mathcal{W}$. After intersecting $\mathcal{W}^{biZar}$ with $\mathcal{A}_\mathbb{C}$ and passing to the geometric generic fiber, we deduce from this together with Lemma 6.2.13 that there exists a finite set of tuples (q_0, H), where $q_0 \in A(K)$ is a torsion point and H is an abelian subvariety of A, such that there exists some point $t \in \mathrm{Tr}'(T'(\mathbb{C}))$ with $\langle W_{K'} \rangle_{\mathbb{C},geo} = t + (q_0 + H)_{K'}$. Here, (T', Tr') is the $K'/\mathbb{C}$-trace of $A_{K'}$.

First of all, $\mathrm{Tr}' : T'_{K'} \to A_{K'}$ is the base change of a homomorphism $T'_{K\mathbb{C}} \to A_{K\mathbb{C}}$ by Theorem 2.4 in [170], because all torsion points of domain and codomain are $K\mathbb{C}$-rational. Hence, (T', Tr') is equal to the base change of the $K\mathbb{C}/\mathbb{C}$-trace of $A_{K\mathbb{C}}$. Furthermore, this latter trace is equal to $(T_\mathbb{C}, \mathrm{Tr}_{K\mathbb{C}})$ by Theorem 6.8 in [33]. It follows that $T' = T_\mathbb{C}$ and $\mathrm{Tr}' = \mathrm{Tr}_{K'}$.

Since it is $\mathbb{C}$-geodesic-optimal for $V_{K'}$ in $A_{K'}$, $W_{K'}$ itself must be equal to an irreducible component of $(t + (q_0 + H)_{K'}) \cap V_{K'}$.

We now want to show that we can take t to be the image of the base change of a $\bar{\mathbb{Q}}$-rational point of T. Indeed, the image of any point in $X = \mathrm{Tr}_{K'}^{-1}(W_{K'} + (-q_0 + H)_{K'})$ that is the base change of a $\mathbb{C}$-rational point of $T_\mathbb{C}$ can be chosen as t. The finite union of subvarieties X is equal to the base change of $\mathrm{Tr}^{-1}(W + (-q_0 + H)) \subset T_K$. On the other hand, one can see that X is equal to $\mathrm{Tr}_{K'}^{-1}(t + H_{K'})$. Since Tr is a homomorphism and every algebraic subgroup of $T_{K'}$ is the base change of an algebraic subgroup of T, this means that X is the base change of a union of translates of an abelian subvariety of $T_\mathbb{C}$ by a point in $T_\mathbb{C}(\mathbb{C})$. Since $\mathbb{C} \cap K = \bar{\mathbb{Q}}$, it follows from Corollaire 4.8.11 in [64] that X is equal to the base change of a union of algebraic subvarieties of T and t can be chosen as the image of the base change of a point of $T(\bar{\mathbb{Q}})$. If we denote this point also by t, we have that W is an irreducible component of $(t + q_0 + H) \cap V$. Since $(t + q_0 + H)_{K'} = \langle W_{K'} \rangle_{\mathbb{C},geo} \subset \langle W \rangle_{K'}$, it also follows that $t + q_0 + H \subset \langle W \rangle$. $\square$

We now apply Theorem 6.3.1 to our problem.

PROPOSITION 6.3.2. *Let $S \subset A_{g,l}$ be a subvariety of positive dimension. Let $\mathcal{A} = \mathfrak{A}_{g,l} \times_{A_{g,l}} S$ and let A be the geometric generic fiber of $\mathcal{A} \to S$. If $\mathrm{ZP}(B, m, d)$ holds for all quotients B of A by a positive-dimensional abelian subvariety, then $\mathrm{ZP}(A, m, d)$ holds.*

PROOF. We induct on m. Clearly $\mathrm{ZP}(A, 0, d)$ holds for all d.

Let V be a subvariety of A of dimension m and let $W \subset V$ be an optimal subvariety of defect at most d. Let (T, Tr) denote the $K/\bar{\mathbb{Q}}$-trace of A, where K is an algebraic closure of $\bar{\mathbb{Q}}(S)$. We have $\mathrm{Tr}(T_K) \neq A$ since $\dim S > 0$.

We apply Theorem 6.3.1. If W satisfies (1), then W is contained in a component of Z and optimal for that component. Furthermore, W has defect at most d, so we are done by induction on m. If W satisfies (2), then W is an irreducible component of $(t + q_0 + H) \cap V$, where $t \in \mathrm{Tr}(T(\bar{\mathbb{Q}}))$ and (q_0, H) lies in a finite set of pairs of torsion points and abelian subvarieties of A that does not depend on W. We can assume H and q_0 fixed. We now quotient out by H. Let $f : A \to A/H$ be the corresponding morphism. We get a subvariety $f(V)$ of A/H and a point w such that $\{w\} = f(W)$. By Lemma 6.2.5, we can find $V_0 \subset V$ open and dense such that $f(V_0)$ is open and dense in $f(V)$ and $f|_{V_0} : V_0 \to f(V_0)$ is smooth of some relative dimension n.

If $W \subset V \backslash V_0$, then W is contained in one of finitely many subvarieties of V of dimension at most $m - 1$ (and of course optimal for that subvariety in A and of defect at most d), so we are done by induction. Hence we can assume that $W \cap V_0 \neq \emptyset$. Since $W \cap V_0$ is an irreducible component of $f|_{V_0}^{-1}(\{w\})$, it follows that $n = \dim(W \cap V_0) = \dim W$.

If $\dim H = 0$, then $W = \{t + q_0\}$. So the singleton $\{t\}$ is contained in some irreducible component V' of $\mathrm{Tr}(T_K) \cap (-q_0 + V)$. It has defect at most d and is optimal for V' in $\mathrm{Tr}(T_K)$. Since $\mathrm{Tr}(T_K) \neq A$, there is an isogeny between $\mathrm{Tr}(T_K)$ and a quotient of A by some positive-dimensional abelian subvariety. We can use our hypothesis and Lemma 6.2.6(2) to deduce that t and therefore W belongs to a finite set. Hence, we can assume that $\dim H > 0$.

By Theorem 6.3.1, we have $t + q_0 + H \subset \langle W \rangle$ and therefore $\langle W \rangle$ contains a translate of a component of $\ker f = H$. It follows from Lemma 6.2.6(1) that $\{w\}$ has defect at most d and is optimal for $f(V)$ in A/H. Now we can use that $\mathrm{ZP}(A/H, m, d)$ holds to deduce that w and hence W as an irreducible component of $f^{-1}(\{w\}) \cap V$ must lie in a finite set. □

6.4. Reduction of the transcendence degree

The following proposition gives us the final reduction to the algebraic case or to what we proved in Proposition 6.3.2.

Proposition 6.4.1. *Let m and d be non-negative integers. Let $K \subset L$ be an extension of algebraically closed fields. Let A be an abelian variety defined over K. If $\mathrm{ZP}(A, m, d)$ holds, then $\mathrm{ZP}(A_L, m, d)$ holds as well.*

Proof. Let V be a subvariety of A_L of dimension at most m. We can find an algebraically closed subfield L_1 of L that has finite transcendence degree over K and a subvariety V_1 of A_{L_1} such that $V = (V_1)_L$. If W is any optimal subvariety for V in A_L, then by Lemma 6.2.10 it is equal to $(W_1)_L$ for an optimal subvariety W_1 for V_1 in A_{L_1} such that $\delta(W_1) = \delta(W)$. Hence, it suffices to prove the proposition under the assumption that L has finite transcendence degree over K.

Arguing by induction on the transcendence degree of L over K, one can see that it is enough to prove our statement when L has transcendence degree 1 over K.

We proceed by induction on m. Clearly $\mathrm{ZP}(A_L, 0, d)$ holds for all d, so, for some positive m, we will deduce $\mathrm{ZP}(A_L, m, d)$ from $\mathrm{ZP}(A_L, m - 1, d)$ and $\mathrm{ZP}(A, m, d)$.

Let V be a subvariety of A_L of dimension m. If $V = V'_L$ for some $V' \subset A$, then we are done by Lemma 6.2.10 and $\mathrm{ZP}(A, m, d)$. We will then assume that this is not the case.

Let V'_L be the smallest subvariety of A_L that is the base change of some $V' \subset A$ and contains V. It exists and has dimension m or $m+1$ by Lemma 6.2.2 but the first case is not possible because it would imply that $V = V'_L$.

Let $W \subset V$ be an optimal subvariety for V in A_L that has defect at most d. We can assume without loss of generality that $W \neq V$.

We let W'_L be the smallest subvariety of A_L that is the base change of some $W' \subset A$ and contains W. By Lemma 6.2.2, we have either $W = W'_L$ or $\dim W'_L = \dim W + 1$.

If $W = W'_L$, then W is contained in $Z'_L \subset V$ for $Z' \subset A$ maximal among all finite unions of subvarieties $Z'' \subset A$ with $Z''_L \subset V$ (and equal to the closure of the union of all such Z''). Since $V \neq V'_L$, the dimension of Z'_L is at most $m-1$. Of course, W is also optimal for the component of Z'_L that contains it and therefore lies in a finite set because $\mathrm{ZP}(A_L, m-1, d)$ holds. We can therefore assume that $W \subsetneq W'_L$ and so $\dim W'_L = \dim W + 1$.

Recall that, by Lemma 6.2.10, an optimal subvariety for V'_L in A_L is the base change of an optimal subvariety for V' in A. Let U'_L be such an optimal subvariety for V'_L in A_L that contains W'_L and satisfies $\delta(U'_L) \leq \delta(W'_L)$. Note that $\langle W \rangle = \langle W'_L \rangle$ and $\langle V \rangle = \langle V'_L \rangle$ because, for instance, $V'_L \subset \langle V \rangle \cap V'_L$ by definition. It follows that $\delta(W'_L) = \delta(W) - 1$, so $\delta(U'_L) \leq d - 1$.

We claim that $U'_L \neq V'_L$. If not, we could deduce that $\delta(V'_L) = \delta(U'_L) \leq \delta(W'_L)$. It would then follow that $\delta(V) = \delta(V'_L) + 1 \leq \delta(W'_L) + 1 = \delta(W)$, which contradicts the optimality of $W \subsetneq V$ for V.

We deduce that $U'_L \subsetneq V'_L$ and hence $U'_L \cap V \subsetneq V$, otherwise $U'_L \supset V$ would contradict the minimality of V'_L. Since $W \subset U'_L \cap V$ and W has defect at most d and is optimal for a component of $U'_L \cap V$ in A_L, it suffices to show that U' and therefore U'_L belongs to a finite set and then we are done as $\dim(U'_L \cap V) < \dim V$ and $\mathrm{ZP}(A_L, m-1, d)$ holds.

It follows from the optimality of U' for V' in A and Proposition 4.5 in [70] that U' is also geodesic-optimal for V' in A. We can apply the results of Rémond in [157] (the connection to geodesic optimality is explained in Section 6 of [70]) to deduce that there exists a finite set of abelian subvarieties of A such that for each geodesic-optimal U for V' in A there exists H in this finite set such that for any $u \in U(K)$ we have $\langle U \rangle_{geo} = u + H$ (and U is an irreducible component of $(u+H) \cap V'$ since it is geodesic-optimal for V').

Since H varies in a finite set, we can assume it fixed and divide out by it. Let $f : A \to A/H$ be the corresponding morphism. We get a subvariety $f(V')$ of A/H and a point $u' \in (A/H)(K)$ such that $\{u'\} = f(U')$.

By Lemma 6.2.5, we can find $V'_0 \subset V'$ open and dense such that $f(V'_0)$ is open and dense in $f(V')$ and $f|_{V'_0} : V'_0 \to f(V'_0)$ is smooth of relative dimension $n = \dim V'_0 - \dim f(V'_0) = \dim V' - \dim f(V')$. If U' is contained in $V' \backslash V'_0$, then U' is contained in one of finitely many subvarieties of V' of dimension at most m. As U' is optimal for V', it is also optimal for that subvariety. We have $\delta(U') \leq \delta(W') = \delta(W) - 1 \leq d-1$. By $\mathrm{ZP}(A, m, d)$, there are then only finitely many possibilities for U'.

Hence we can assume that $U' \cap V'_0 \neq \emptyset$. Since U' is an irreducible component of $(u+H) \cap V'$ (for some $u \in U'(K)$), $U' \cap V'_0$ is then an irreducible component of $(u+H) \cap V'_0 = f|_{V'_0}^{-1}(\{u'\})$. It follows that $n = \dim(U' \cap V'_0) = \dim U'$.

Since we know that $W \neq W'_L$, we have $\dim W' > 0$ and hence $n = \dim U' > 0$.

Note that $\langle U' \rangle$ contains a translate of a component of $\ker f = H$ since $\langle U' \rangle_{geo} = u + H \subset \langle U' \rangle$ (for $u \in U'(K)$ arbitrary). It therefore follows from Lemma 6.2.6(1) that $\{u'\}$ is optimal for $f(V')$ in A/H and has defect at most $d-1$. Furthermore, $f(V')$ is a subvariety of A/H of dimension $\dim V' - n \leq \dim V' - 1 = m$.

Now, there is a homomorphism $A/H \to A$ of algebraic groups with finite kernel, so $\mathrm{ZP}(A/H, m, d)$ holds by Lemma 6.2.6(2). Thus, u' lies in a finite set. As U' is a component of $f^{-1}(\{u'\}) \cap V'$, it lies in a finite set as well. $\square$

6.5. Proof of Theorem 6.1.5

Fix non-negative integers m and d. We argue by induction on the dimension of A. If the dimension of A is at most 1, the statement holds trivially. Let now A be an abelian variety of dimension > 1 over an algebraically closed field K and assume that Theorem 6.1.5 holds for the fixed m and d and all abelian varieties of smaller dimension. In particular, it holds for all quotients of A by abelian subvarieties of positive dimension.

Applying Lemma 6.2.7, we find a subvariety S of $A_{g,l}$ and an embedding of $\bar{\mathbb{Q}}(S)$ into K such that A is isogenous to A'_K, where A' is the generic fiber of $\mathfrak{A}_{g,l} \times_{A_{g,l}} S$.

Moreover, the $K/\bar{\mathbb{Q}}$-traces of A and A'_K are isogenous. Therefore, by Lemma 6.2.6(2), we only need to prove the statement of Theorem 6.1.5 for A'_K.

Proposition 6.4.1 tells us that $\mathrm{ZP}(A'_K, m, d)$ follows from $\mathrm{ZP}(A'_{K'}, m, d)$, where K' is an algebraic closure of $\bar{\mathbb{Q}}(S)$.

If $\dim S = 0$, we have nothing to do. We then assume that S has positive dimension.

By the inductive hypothesis we know that for all quotients B of $A'_{K'}$ by a positive-dimensional abelian subvariety with $K'/\bar{\mathbb{Q}}$-trace (T_B, Tr_B), the implication

$$\mathrm{ZP}(T_B, m, d) \implies \mathrm{ZP}(B, m, d)$$

holds.

Note that the $K/\bar{\mathbb{Q}}$-trace of A'_K is equal to the base change of the $K'/\bar{\mathbb{Q}}$-trace $(T_{A'_{K'}}, \mathrm{Tr}_{A'_{K'}})$ of $A'_{K'}$ by Theorem 6.4(3) in [33]. If we know that $\mathrm{ZP}(T_{A'_{K'}}, m, d)$ holds, then, for all B quotients of $A'_{K'}$, since there exists a homomorphism of algebraic groups with finite kernel from T_B to $T_{A'_{K'}}$, $\mathrm{ZP}(T_B, m, d)$ holds as well because of Lemma 6.2.6(2).

The inductive hypothesis then tells us that $\mathrm{ZP}(B, m, d)$ holds for all quotients B of $A'_{K'}$ by a positive-dimensional abelian subvariety and thus, by Proposition 6.3.2, $\mathrm{ZP}(A'_{K'}, m, d)$ holds as we wanted to prove. $\square$

6.6. Proof of Theorem 6.1.9

We show that the hypotheses in Theorem 6.1.9 imply the following claim for every quotient A' of A:

Claim 1. *Every subvariety V of A' of dimension at most m contains at most finitely many optimal singletons (for V in A') of defect at most d.*

As every quotient of A admits a homomorphism of algebraic groups with finite kernel to A, we can deduce as in the proof of Lemma 6.2.6(2) that it suffices to prove Claim 1 for A itself.

Let therefore V be a subvariety of A of dimension at most m. We show the following claim by induction on $j \in \{0, \dots, \dim V\}$:

CLAIM 2. *The optimal singletons for V in A of defect at most d are contained in a finite union of subvarieties of V of dimension at most* $\dim V - j$.

This is obvious for $j = 0$.

Suppose that Claim 2 has been proven for some $j < \dim V$. Let W be one of the finitely many subvarieties of V of dimension at most $\dim V - j$ that contain the optimal singletons for V in A of defect at most d. We can assume without loss of generality that $\dim W = \dim V - j$.

Any optimal singleton for V in A that is contained in W is also optimal for W in A. We want to show that the optimal singletons for W in A of defect at most d are contained in a proper closed subset of W. This will establish Claim 2 for $j+1$.

Translating W by a torsion point sends optimal singletons (for W in A) to optimal singletons of the same defect, so we can assume without loss of generality that $B := \langle W \rangle$ is an abelian subvariety of A.

If $\{p\} \subset W$ is an optimal singleton for W in A of defect at most d, then $\langle \{p\} \rangle \subset B$. Since $\dim W = \dim V - j > 0$, we have that $\{p\} \subsetneq W$ and therefore $\delta(\{p\}) = \dim\langle \{p\} \rangle < \delta(W) = \dim B - \dim W$. It follows that the codimension of $\langle \{p\} \rangle$ in B is greater than or equal to $k := \max\{\dim B - d, \dim W + 1\}$. So the optimal singletons for W in A of defect at most d are contained in $W \cap B^{[k]}$.

As $B = \langle W \rangle$, no proper special subvariety of B can contain W. It then follows from Conjecture 6.1.8 for B, d, and W that $W \cap B^{[k]}$ is not dense in W. Together with the above, this implies that the optimal singletons for W in A of defect at most d are contained in a proper closed subset of W as desired. This establishes Claim 2 by induction.

Now taking $j = \dim V$ shows that the number of optimal singletons for V in A of defect at most d is finite. This proves Claim 1. Theorem 6.1.9 now follows from the following theorem:

THEOREM 6.6.1. *Let m and d be non-negative integers and suppose that every subvariety of dimension at most m of a quotient of A contains at most finitely many optimal singletons of defect at most d. Then every subvariety of A of dimension at most m contains at most finitely many optimal subvarieties of defect at most d.*

PROOF. The hypotheses imply that after fixing an arbitrary field of definition that is finitely generated over $\mathbb{Q}$, every quotient of A satisfies LGO_d^m as defined in Definition 8.1 in [70]. Theorem 6.6.1 then follows from Theorem 9.8(i) in [70]. $\square$

6.7. The Zilber-Pink conjecture for subvarieties of defect ≤ 1

THEOREM 6.7.1. *Let A be an abelian variety over $\bar{\mathbb{Q}}$ and $V \subset A$ a subvariety. Then V contains at most finitely many optimal subvarieties of defect at most* 1.

The following proof was suggested to the authors by Philipp Habegger.

PROOF. Let $\{p\} \subset V$ be an optimal singleton for V in A, contained in a torsion coset of dimension at most 1. By Proposition 4.5 in [70], $\{p\}$ is geodesic-optimal for V in A and therefore not contained in a coset of positive dimension that is contained in V. By the Theorem in [66] with $s = \dim A - 1$, the height of p with respect to any fixed symmetric ample line bundle on A is then bounded.

It then follows from Proposition 9.7 in [70] that (in the notation of [70]) $LGO_1(V)$ is satisfied after fixing a number field over which V and A are defined. As V was arbitrary, this implies that every abelian variety A over $\bar{\mathbb{Q}}$ satisfies LGO_1^m for all integers $m \geq 0$; see Definition 8.1 in [70] for the definitions of $LGO_d(V)$ and LGO_d^m. The claim then follows from Theorem 9.8(i) in [70]. $\square$

Appendices

APPENDIX A

Generalized Vojta-Rémond inequality

> Il y a maintenant en France dans chaque village un flambeau allumé, le maître d'école, et une bouche qui souffle dessus, le curé.
>
> V. Hugo, *Histoire d'un crime*

A.1. Introduction

Let $m \geq 2$ be an integer and let $X_1, \ldots, X_m$ be a family of irreducible positive-dimensional projective varieties, defined over $\bar{\mathbb{Q}}$. We wish to extend Rémond's results in [154] to the case of an algebraic point $x = (x_1, \ldots, x_m)$ in the product $X_1 \times \cdots \times X_m$. This chapter is a further generalization of a generalization of these results by Thomas Ange. It draws heavily on a written account of the original generalization by Ange [5].

In Chapter 4, we apply our generalized Vojta inequality to a relative version of the Mordell-Lang problem in an abelian scheme $\mathcal{A} \xrightarrow{\pi} S$, where S is an irreducible variety and everything is defined over $\bar{\mathbb{Q}}$. In the problem, one fixes an abelian variety A_0, defined over $\bar{\mathbb{Q}}$, a finite rank subgroup $\Gamma \subset A_0(\bar{\mathbb{Q}})$, and an irreducible subvariety $\mathcal{V} \subset \mathcal{A}$ and studies the points $p \in \mathcal{V}$ of the form $\phi(\gamma)$ for an isogeny $\phi : A_0 \to \mathcal{A}_{\pi(p)}$, $\mathcal{A}_{\pi(p)}$ denoting the fiber of the abelian scheme over $\pi(p)$, and $\gamma \in \Gamma$.

In this application, it is crucial that we allow the X_i to lie in different fibers of the abelian scheme. If the abelian scheme $\mathcal{A}$ is constant, an analogue of the intended height bound has been obtained by von Buhren in [188]. In his case, the generalized Vojta inequality from [154], where $X_1 = X_2 = \cdots = X_m = X$, was sufficient, however for our intended application, it is necessary to allow the X_i to be different.

Let us recall the hypotheses which come into play. We use (almost) the same notation as in [154] and we refer to that article for the history of Vojta's inequality.

For an m-tuple $a = (a_1, \ldots, a_m)$ of positive integers, we write

$$\mathcal{N}_a = \bigotimes_{i=1}^{m} p_i^* \mathcal{L}_i^{\otimes a_i},$$

where $\mathcal{L}_i$ is a fixed very ample line bundle on X_i and $p_i : X_1 \times \cdots \times X_m \to X_i$ is the natural projection. We fix a non-empty open subset $U^0 \subset X_1 \times \cdots \times X_m$ and relate a to an irreducible projective variety $\mathcal{X}$, provided with an open immersion $U^0 \subset \mathcal{X}$

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and a proper morphism $\pi : \mathcal{X} \to X_1 \times \cdots \times X_m$ such that $\pi|_{U^0} = \mathrm{id}_{U^0}$, as well as to a nef line bundle $\mathcal{M}$ on $\mathcal{X}$ which satisfies some further conditions, specified below.

We assume that there exists a very ample line bundle $\mathcal{P}$ on $\mathcal{X}$, an injection $\mathcal{P} \hookrightarrow \mathcal{N}_a^{\otimes t_1}$ which induces an isomorphism on U^0 and a system of homogeneous coordinates Ξ for $\mathcal{P}$ which are (by means of the aforementioned injection) monomials of multidegree $t_1 a$ in the homogeneous coordinates $W^{(i)} \subset \Gamma(X_i, \mathcal{L}_i)$, fixed in advance (we denote $\pi^*\mathcal{N}_a$ also by $\mathcal{N}_a$ and identify $p_i^* W^{(i)}$ and $\pi^* p_i^* W^{(i)}$ with $W^{(i)}$). By (a system of) homogeneous coordinates for a very ample line bundle, we mean the set of pull-backs of the homogeneous coordinates on some $\mathbb{P}^{N'}$ under a closed embedding into $\mathbb{P}^{N'}$ that is associated to that line bundle.

We also assume that there exists an injection $(\mathcal{P} \otimes \mathcal{M}^{\otimes -1}) \hookrightarrow \mathcal{N}_a^{\otimes t_2}$ which induces an isomorphism on U^0 and that $\mathcal{P} \otimes \mathcal{M}^{\otimes -1}$ is generated by a family Z of M global sections on $\mathcal{X}$ which are polynomials $P_1, \ldots, P_M$ of multidegree $t_2 a$ in the $W^{(i)}$ such that the height of the family of coefficients of all these polynomials is at most $\sum_i a_i \delta_i$. Recall that the height of a finite subset of $\bar{\mathbb{Q}}$ is defined by considering it as a point in a suitable projective space and that on projective space, the height is defined as in Definition 1.5.4 in [20] by use of the maximum norm at the infinite places. The height of any polynomial is defined as the height of the family of its coefficients.

The integer parameters t_1, t_2, M and the real parameters $\delta_1, \ldots, \delta_m$ (all at least 1) are fixed independently of the triple $(a, \mathcal{X}, \mathcal{M})$. This triple permits to define the following two notions of height for an algebraic point $x \in U^0(\bar{\mathbb{Q}})$:

$$h_{\mathcal{M}}(x) = h(\Xi(x)) - h(Z(x)),$$

$$h_{\mathcal{N}_a}(x) = a_1 h\left(W^{(1)}(x)\right) + \cdots + a_m h\left(W^{(m)}(x)\right).$$

Our goal is to prove an inequality among these two numbers under certain assumptions about the intersection numbers of $\mathcal{M}$. Let therefore $\theta \geq 1$ and $\omega \geq -1$ be two integer parameters and set (with $\omega' = 3 + \omega$)

$$\Lambda = \theta (2 t_1 u_0)^{u_0} \left(\max_{1 \leq i \leq m} N_i + 1 \right) \prod_{i=1}^{m} \deg(X_i),$$

$$\psi(u) = \prod_{j=u+1}^{u_0} (\omega' j + 1),$$

$$c_1 = c_2 = \Lambda^{\psi(0)},$$

$$c_3^{(i)} = \Lambda^{2\psi(0)} (M t_2)^{u_0} (h(X_i) + \delta_i) \quad (i = 1, \ldots, m),$$

where $u_0 = \dim(X_1) + \cdots + \dim(X_m)$, $N_i + 1 = \# W^{(i)}$, and the degrees and heights are computed with respect to the embeddings given by the $W^{(i)}$. We use here the (normalized) height of a subvariety of projective space as defined in [26] (via Arakelov theory) or [131] (via Chow forms). The two definitions yield the same height by Théorème 3 in [173].

The following theorem therefore generalizes Théorème 1.2 in [154].

Theorem A.1.1. *Let $x \in U^0(\bar{\mathbb{Q}})$ be an algebraic point and $(a, \mathcal{X}, \mathcal{M})$ a triple as defined above. Suppose that, for every subproduct of the form $Y = Y_1 \times \cdots \times Y_m$,* *where $Y_i \subset X_i$ is an irreducible subvariety that contains x_i, we have the following estimate*

$$(\mathcal{M}^{\cdot \dim(Y)} \cdot \mathcal{Y}) \geq \theta^{-1} \prod_{i=1}^{m} (\deg(Y_i))^{-\omega} a_i^{\dim(Y_i)},$$

where $\mathcal{Y}$ denotes the closure of $\pi^{-1}(Y \cap U^0)$ in $\mathcal{X}$. Then we have

$$h_{\mathcal{N}_a}(x) \leq c_1 h_{\mathcal{M}}(x)$$

if furthermore $c_2 a_{i+1} \leq a_i$ for every $i < m$ and $c_3^{(i)} \leq h\left(W^{(i)}(x_i)\right)$ for every $i \leq m$.

The fact that for each i there is a different constant $c_3^{(i)}$ is the main difference with Ange's work, where there is just one δ instead of $\delta_1, \ldots, \delta_m$ (in our set-up, δ can be taken as $\max_{1 \leq i \leq m} \delta_i$) and there is just one constant c_3 defined as

$$\Lambda^{2\psi(0)}(Mt_2)^{u_0} \max\left\{\max_{1 \leq i \leq m} h(X_i), \delta\right\}.$$

The condition that x_i has large height then reads $c_3 \leq h\left(W^{(i)}(x_i)\right)$. Ange's inequality is a direct generalization of Rémond's inequality (up to the slightly different definitions of Λ and c_1).

If we set $\delta = \max_{1 \leq i \leq m} \delta_i$, then the inequality $c_3^{(i)} \leq h\left(W^{(i)}(x_i)\right)$ follows from $2c_3 \leq h\left(W^{(i)}(x_i)\right)$, so Theorem A.1.1 really is a generalization of Rémond's work (up to the factor 2 and the slightly different definitions of Λ and c_1). In the application in Chapter 4, the fact that $c_3^{(i)}$ depends only on $h(X_i)$ and δ_i and not on $h(X_j)$ or δ_j ($j \neq i$) is crucial. Ange's version of the inequality is therefore not sufficient for the application.

Naturally, we follow the proof in [154] very closely with some minor changes: Firstly, the term $12u\psi(u)$ that appears in the last equation of [154] should be replaced by $4\omega' u\psi(u)$; that is why we do not use Lemme 5.4 from [154] and define Λ slightly differently. Secondly, Corollaire 5.1 in [154] does not apply if $x_j^{(i)} = 0$, which means that Corollaire 3.2 in [154] has to be made more precise. Thirdly, in the last inequality in the proof of Proposition 4.2 in [154], a term bounding the contribution of the infinite places when the P_i are raised to the d-th power is missing. Fourthly, the factor 8 in the upper bound $8(N+1)D_i \log(N+1)D_i$ for $\log 2f_2(u_i, D_i)$ given in the proof of Proposition 5.3 in [154] has to be increased. Fifthly, we had to impose that $\mathcal{M}$ is nef in order to be able to translate the lower bound on its top self-intersection number into a lower bound for the dimension of a space of global sections.

A.2. Reduction to a minimal subproduct

We first consider a subproduct $Y = Y_1 \times \cdots \times Y_m$ of minimal total dimension $u = u_1 + \cdots + u_m$, satisfying the following conditions:

(i) $x_i \in Y_i(\bar{\mathbb{Q}})$ for all $1 \leq i \leq m$;
(ii) $d_i \leq \deg(X_i)\Lambda^{\psi(u)-1}$ for all $1 \leq i \leq m$;
(iii) $\prod_{i=1}^{m} d_i \leq (\prod_{i=1}^{m} \deg(X_i))\Lambda^{\psi(u)-1}$;
(iv) $\sum_{i=1}^{m} a_i(h_i + \delta_i) \leq 2^{-1}\Lambda^{2\psi(u)}(Mt_2)^{u_0-u} \sum_{i=1}^{m} (a_i(h(X_i) + \delta_i))$,

where $u_i = \dim(Y_i)$, $d_i = \deg(Y_i)$, and $h_i = h(Y_i)$ (in the projective embedding defined by $W^{(i)}$). Such a subproduct certainly exists since $X_1 \times \cdots \times X_m$ satisfies these conditions. Furthermore, we have $u > 0$ since otherwise $Y = \{x\}$ and therefore

$$\sum_{i=1}^{m} a_i c_3^{(i)} \leq h_{\mathcal{N}_a}(x) \leq \sum_{i=1}^{m} a_i h_i \leq \frac{1}{2} \sum_{i=1}^{m} a_i c_3^{(i)}.$$

We use the definition of an adapted projective embedding on p. 466 of [154]. By Proposition 2.2 in [154], we may define an embedding adapted to the subvariety Y_i of X_i by putting

$$V_j^{(i)} = \sum_{k=0}^{N_i} M_{jk}^{(i)} W_k^{(i)}$$

with $M^{(i)} \in \mathrm{GL}_{N_i+1}(\mathbb{Q})$, where the coefficients of the matrix $M^{(i)}$ are integers and bounded by $\max\left(1, \frac{d_i}{2}\right)$ in absolute value, at least if $Y_i \neq \mathbb{P}^{N_i}$. If $Y_i = \mathbb{P}^{N_i}$, then the notion of an adapted projective embedding is not defined in [154], but we may set $V_j^{(i)} = W_j^{(i)}$ ($j = 0, \ldots, N_i$) and check that all the assertions about adapted embeddings made in this chapter also hold true in this case.

We now prove the equivalent of Proposition 3.1 in [154], introducing

$$\Lambda_h = \sum_{i=1}^{m} a_i(h_i + \delta_i + d_i(u_i + 1)\log 2d_i(N_i + 1)),$$

which we will prove to verify

$$\Lambda_h < \Lambda^{2\psi(u)}(Mt_2)^{u_0-u} \sum_{i=1}^{m} (a_i(h(X_i) + \delta_i)) = \Lambda^{2\psi(u)-2\psi(0)}(Mt_2)^{-u} \sum_{i=1}^{m} a_i c_3^{(i)}. \tag{A.2.1}$$

In order to show this inequality (given condition (iv) from above), it suffices to show that

$$\sum_{i=1}^{m} a_i d_i(u_i + 1)\log 2d_i(N_i + 1) < 2^{-1}\Lambda^{2\psi(u)}(Mt_2)^{u_0-u} \sum_{i=1}^{m} (a_i(h(X_i) + \delta_i))$$

or even $\sum_{i=1}^{m} d_i(u_i + 1)\log 2d_i(N_i + 1) < 2^{-1}\Lambda^{2\psi(u)}(Mt_2)^{u_0-u}$. But since $\log 2d_i(N_i + 1) < 2d_i(N_i + 1)$, it follows from (ii) that the left-hand side is at most

$$(u + m)\Lambda^{2\psi(u)-2} \max_{1\leq i\leq m} \{2\deg(X_i)^2(N_i + 1)\} < 2^{-1}\Lambda^{2\psi(u)}$$

and now the claim is obvious.

Proposition A.2.1. *There does not exist any pair (l, U) such that $1 \leq l \leq m$ and $U\left(V^{(l)}\right)$ is a homogeneous polynomial in the first adapted coordinates $V_0^{(l)}$, …, $V_{u_l}^{(l)}$ satisfying*

(a) $U\left(V^{(l)}\right)(x_l) = 0$;
(b) *U is not the zero polynomial;*
(c) $\deg(U) \leq \Lambda^{\omega' u\psi(u)}$;
(d) $a_l h(U) \leq \Lambda^{2\psi(u-1)-2\psi(u)}\left(\frac{Mt_2}{4d_l}\right)\Lambda_h$.

Proof. We assume the contrary and define Y_l' as an irreducible component containing x_l of the subvariety of Y_l defined by the equation $U\left(V^{(l)}\right)=0$ and we verify that the subproduct Y' obtained by replacing Y_l by Y_l' in Y contradicts the minimality of the latter. We have $Y'=Y_1'\times\cdots\times Y_m'$ with $Y_i'=Y_i$ for all $i\neq l$. By (a), condition (i) holds for Y'. By (b) and the definition of an adapted embedding, Y' is a proper subvariety of Y.

The polynomial $U\left(V^{(l)}\right)$ corresponds by means of $M^{(l)}$ to a polynomial $U'\left(W^{(l)}\right)$, where $\deg(U')=\deg(U)$ and

$$h(U')\leq h(U)+\deg(U)\log(N_l+1)\max\left(1,\frac{d_l}{2}\right)+\log\binom{\deg(U)+u_l}{\deg(U)}.$$

As

$$\binom{\deg(U)+u_l}{\deg(U)}=\prod_{i=1}^{\deg(U)}\left(1+\frac{u_l}{i}\right)\leq(1+u_l)^{\deg(U)},$$

it follows that

$$h(U')\leq h(U)+\deg(U)\log d_l(N_l+1)(u_l+1). \tag{A.2.2}$$

The (arithmetic as well as geometric) theorems of Bézout yield

$$\deg(Y_l')\leq\deg(U')d_l$$

and

$$h(Y_l')\leq\deg(U')h_l+d_l\left(h(U')+\sqrt{N_l}\right).$$

For the arithmetic Bézout theorem, we use Théorème 3.4 and Corollaire 3.6 from [152], where the modified height h_m used there can be bounded thanks to Lemme 5.2 in [153]. Together with (c), the first line implies that $\deg(Y_l')\leq d_l\Lambda^{\psi(u-1)-\psi(u)}$ since by definition $\psi(u-1)=(\omega'u+1)\psi(u)$. This shows that Y' satisfies conditions (ii) and (iii).

From the second line together with (A.2.2), (c), and (d), we deduce that

$$\begin{aligned}\sum_{i=1}^m a_i(h(Y_i')+\delta_i)&\leq d_la_lh(U)+d_la_l\Lambda^{\omega'u\psi(u)}\log d_l(N_l+1)(u_l+1)\\ &+d_la_l\sqrt{N_l}+\Lambda^{\omega'u\psi(u)}\sum_{i=1}^m a_i(h_i+\delta_i)\leq d_la_lh(U)+3\Lambda^{\omega'u\psi(u)}\Lambda_h\\ &\leq\Lambda^{2\psi(u-1)-2\psi(u)}\left(\frac{Mt_2}{4}\right)\Lambda_h+3\Lambda^{\omega'u\psi(u)}\Lambda_h.\end{aligned}$$

Finally, we have $3\Lambda^{\omega'u\psi(u)}\leq\Lambda^{2\omega'u\psi(u)}\left(\frac{Mt_2}{4}\right)=\Lambda^{2\psi(u-1)-2\psi(u)}\left(\frac{Mt_2}{4}\right)$. It then follows from (A.2.1) that Y' satisfies condition (iv) as well and we get the desired contradiction. □

We proceed to deduce from this an equivalent of Corollaire 3.2 in [154] (with a modification of the last assertion). Let us mention that by Lemme 2.3 in [154], there exist polynomial relations

$$P_j^{(i)}\left(V_0^{(i)},\ldots,V_{u_i}^{(i)},V_j^{(i)}\right)=Q_j^{(i)}\left(V_0^{(i)},\ldots,V_{u_i}^{(i)},W_j^{(i)}\right)=0\text{ in }\Gamma\left(Y_i,\mathcal{L}_i^{\otimes d_i}\right)$$

for all $1 \leq i \leq m$ and all $0 \leq j \leq N_i$. The polynomials $P_j^{(i)}(T)$ and $Q_j^{(i)}(T)$ are homogeneous of degrees d_i, monic in their last variable T_{u_i+1}, and equal to a power of an irreducible polynomial (we denote the corresponding exponent for $Q_j^{(i)}$ by $b_{i,j}$). Furthermore, we know from the same lemma that the height of the family B_i of all the coefficients of the $P_j^{(i)}$ and the $Q_j^{(i)}$ for fixed i can be estimated from above as

$$h(B_i) \leq h_i + d_i(u_i+1)\log d_i(N_i+1). \tag{A.2.3}$$

Corollary A.2.2. *For every index $1 \leq i \leq m$, we have that*

(1) the morphism $\rho_i : Y_i \to \mathbb{P}^{u_i}$, defined by the first adapted coordinates $V_0^{(i)}$, $\dots, V_{u_i}^{(i)}$, is finite, surjective, and étale at $x_i \in Y_i(\bar{\mathbb{Q}})$;
(2) $V_0^{(i)}(x_i) \neq 0$;
(3) for every index $0 \leq j \leq N_i$ such that $W_j^{(i)} \neq 0$ in $\Gamma(Y_i, \mathcal{L}_i)$, we have

$$W_j^{(i)} \frac{\partial^{b_{i,j}} Q_j^{(i)}}{\partial T_{u_i+1}^{b_{i,j}}} \left(1, \frac{V_1^{(i)}}{V_0^{(i)}}, \dots, \frac{V_{u_i}^{(i)}}{V_0^{(i)}}, \frac{W_j^{(i)}}{V_0^{(i)}}\right)(x_i) \neq 0.$$

Proof. That the morphism ρ_i is finite and surjective follows from the definition of adapted embeddings (see [154], Section 2.1). If one of the three assertions were not true, we could construct a pair $\left(i, U\left(V^{(i)}\right)\right)$ that would contradict Proposition A.2.1 with $\deg(U) \leq 2d_i^2$ and $h(U) \leq 6N_i d_i^3 + 2d_i h(B_i)$.

We refer to Corollaire 3.2 in [154] for the proof – in the case that $W_j^{(i)} \neq 0$, $W_j^{(i)}(x_i) = 0$, it suffices to take $U\left(V^{(i)}\right) = Q_j^{(i)}\left(V_0^{(i)}, \dots, V_{u_i}^{(i)}, 0\right)$. Note that $P_{u_i+1}^{(i)}$ is not only a power of an irreducible polynomial, but in fact irreducible, since its degree is equal to the degree of Y_i, which is also equal to the degree of any irreducible factor of $P_{u_i+1}^{(i)}$. Hence, its discriminant does not vanish identically. That the morphism ρ_i is étale at x_i is proven in the same way as in the proof of Lemme 4.3 in [150]. $\square$

A.3. Constructing a section of small height

Following Section 4 of [154], we set

$$\epsilon = \frac{1}{2u\theta} \frac{1}{(t_1 m)^u} \prod_{i=1}^{m} d_i^{-1-\omega}$$

and define a family of sections $Z'_d \subset \Gamma(\mathcal{X}, \mathcal{M}^{\otimes -d} \otimes \mathcal{P}^{\otimes d} \otimes \mathcal{N}_a^{\otimes d\epsilon})$ of cardinality $M' = M(N_1+1)\cdots(N_m+1)$ for every $d \in \epsilon^{-1}\mathbb{N} \subset \mathbb{N}$ by

$$Z'_d = \left\{\zeta^{\otimes d} \otimes \left(W_{j_1}^{(1)}\right)^{\otimes d\epsilon a_1} \otimes \cdots \otimes \left(W_{j_m}^{(m)}\right)^{\otimes d\epsilon a_m}; \zeta \in Z, W_{j_i}^{(i)} \in W^{(i)}\right\}.$$

The proof of Proposition 4.1 in [154] then goes through without any major modifications (given that $\mathcal{M}$ is nef, see the proof of Proposition A.3.1 below). It yields a natural number d_0 that we choose sufficiently large so that for each natural number $d \geq d_0$ there exists a basis of $\Gamma(\mathcal{Y}, \mathcal{P}^{\otimes d})$ that consists of monomials of degree d in the elements of Ξ. We obtain the following equivalent of Proposition 4.2 in [154].

PROPOSITION A.3.1. *For $d \in \epsilon^{-1}\mathbb{N} \cap d_0\mathbb{N}$, we write $\mathcal{Q}_d = \mathcal{M}^{\otimes d} \otimes \mathcal{N}_a^{\otimes -d\epsilon}$ and fix a basis of $\Gamma(\mathcal{Y}, \mathcal{P}^{\otimes d})$ that consists of monomials in the sections Ξ of degree d.*

Then there exists a section $0 \neq s \in \Gamma(\mathcal{Y}, \mathcal{Q}_d)$ such that the height of s, defined as the height of the family of coefficients of the sections $s \otimes Z'_d$ with respect to the fixed basis, satisfies

$$h(s) \leq \frac{2M'd}{u\epsilon}(t_1 + 2t_2 + \epsilon)\Lambda_h + o(d).$$

PROOF. The dimension estimate

$$\dim \Gamma(\mathcal{Y}, \mathcal{Q}_d) \geq \frac{d^u}{4\theta u!} \prod_{i=1}^{m} d_i^{-\omega} a_i^{u_i} + O(d^{u-1})$$

given in Proposition 4.1 in [154] is still valid since the intersection numbers are formally the same. Here, we need however that $\mathcal{M}$ is nef in order to translate the lower bound for its top self-intersection number into a lower bound for the dimension of a space of global sections through the asymptotic Riemann-Roch theorem (see [78], Theorem VI.2.15).

In the Faltings complex on $\mathcal{Y}$ defined by the family Z'_d of cardinality M'

$$0 \to \mathcal{Q}_d \to \left(\mathcal{P}^{\otimes d}\right)^{\oplus M'} \to \left(\mathcal{N}_a^{\otimes d(t_1+t_2+\epsilon)}\right)^{\oplus (M')^2},$$

the image of $\Gamma(\mathcal{Y}, \mathcal{Q}_d)$ in $F = \Gamma(\mathcal{Y}, \mathcal{P}^{\otimes d})^{M'}$ coincides with the kernel of a family of linear forms in the coordinates with respect to the fixed basis. This family can be chosen such that the coefficients of the linear forms lie in a number field that is independent of d, and that moreover the height of the set of all coefficients is at most $d(t_1 + 2t_2 + \epsilon)\Lambda_h + o(d)$: In order to show this, we follow the proof of Proposition 4.2 in [154] by applying Lemme 2.5 from [154] with $n_i = N_i$ and use (A.2.3) to bound $h(B_i)$. Note that when estimating $h(P_1^d, \ldots, P_M^d)$ as in the proof of Proposition 4.2, one obtains by well-known height estimates an upper bound of

$$d \sum_{i=1}^{m} a_i \delta_i + dt_2 \sum_{i=1}^{m} a_i \log(N_i + 1)$$

(the second summand, coming from the infinite places, is missing in [154]).

Furthermore, the injection $\mathcal{P}^{\otimes d} \hookrightarrow \mathcal{N}_a^{\otimes dt_1}$ yields that

$$\dim F \leq M' \prod_{i=1}^{m} \frac{d_i}{u_i!}(dt_1 a_i)^{u_i} + o(d^u)$$

and so $\log \dim F = o(d)$. Hence, the Dirichlet exponent of the system can be estimated as

$$\frac{\dim F}{\dim \Gamma(\mathcal{Y}, \mathcal{Q}_d)} \leq \frac{2M'}{u\epsilon} + o(1)$$

and the proposition follows from the Siegel lemma (Lemme 2.6 in [154]). $\square$

A.4. The index is small

We now replace $\mathcal{Y}$ by a sufficiently small open subset of $\mathcal{Y}$ that contains x. According to Corollary A.2.2, we can in particular assume that each section $V_0^{(i)}$ vanishes nowhere on this subset and suppose that the sheaf of differentials $\Omega_{\mathcal{Y}/\bar{\mathbb{Q}}}$ is generated by the differentials of the $V_j^{(i)}/V_0^{(i)}$ ($i=1,\dots,m$, $1 \le j \le u_i$). We can furthermore suppose that $\mathcal{P}$, $\mathcal{M}$, and $\mathcal{N}_a$ all can be trivialized over this subset.

We fix an isomorphism $\mathcal{Q}_d \simeq \mathcal{O}_{\mathcal{Y}}$ and consider the index σ (as defined in Section 5.2 of [154]) of the section $s = s_d \in \Gamma(\mathcal{Y}, \mathcal{Q}_d)$, which was constructed in the preceding proposition, with respect to the weight $dt_1 a$ in x.

Lemma A.4.1. *With notations as above, we have*

$$\sigma \le (4t_1 \max_i d_i(N_i+1))^{-1}\epsilon$$

for $d \in \epsilon^{-1}\mathbb{N} \cap d_0\mathbb{N}$ sufficiently large.

Proof. We assume that the inequality is false and derive a contradiction. We can estimate

$$\begin{aligned}\sigma \prod_{i=1}^m d_i^{-1} &\ge (4t_1 \max_i d_i(N_i+1))^{-1}\epsilon \prod_{i=1}^m d_i^{-1}\\ &\ge (8u\theta t_1^{u+1} m^u \max_i (N_i+1))^{-1} \prod_{i=1}^m d_i^{-\omega'}.\end{aligned}$$

It then follows from (iii) that

$$\sigma \prod_{i=1}^m d_i^{-1} \ge (8u\theta t_1^{u+1} m^u \max_i (N_i+1))^{-1} \prod_{i=1}^m (\deg X_i)^{-\omega'} \Lambda^{-\omega'(\psi(u)-1)}$$

and hence $\sigma \prod_{i=1}^m d_i^{-1} \ge \sigma_0 = m\Lambda^{-\omega'\psi(u)}$.

Then, we can construct a non-zero multihomogeneous polynomial $G(V)$ of multidegree $dt_1(d_1\cdots d_m)a$ in the adapted coordinates $V_j^{(i)}$, $0 \le j \le u_i$, of height bounded by

$$h(G) \le (d_1 \cdots d_m)\left(h(s_d) + dt_1 \sum_{i=1}^m a_i(h(B_i) + \log(2(u_i+1)))\right) + o(d)$$

and of index at least σ in $\rho(x)$ with respect to the weight $dt_1 a$, where $\rho = (\rho_1 \circ p_1|_Y, \dots, \rho_m \circ p_m|_Y)$.

For this, we choose $\zeta' \in Z_d'$ which does not vanish at x. We write $s_d \otimes \zeta' = \alpha\left(\left(V_0^{(1)}\right)^{\otimes dt_1 a_1} \otimes \cdots \otimes \left(V_0^{(m)}\right)^{\otimes dt_1 a_m}\right)$, where α is a polynomial in the $W_j^{(i)}/V_0^{(i)}$ with coefficients in $\bar{\mathbb{Q}}$. Consider the norm $N(\alpha)$ of α with respect to the field extension $\bar{\mathbb{Q}}(\mathcal{Y})/L$, where L is the subfield of $\bar{\mathbb{Q}}(\mathcal{Y})$ generated by the $V_j^{(i)}/V_0^{(i)}$ ($j = 1,\dots,u_i$, $i = 1,\dots,m$). We can take

$$G = \left(V_0^{(1)}\right)^{dt_1(d_1\cdots d_m)a_1} \cdots \left(V_0^{(m)}\right)^{dt_1(d_1\cdots d_m)a_m} N(\alpha).$$

On the one hand, this is a quotient of multihomogeneous elements of $R = \bar{\mathbb{Q}}[V_j^{(i)}; 1 \le i \le m, 0 \le j \le u_i]$. As such it has a multidegree, which is exactly $dt_1(d_1 \cdots d_m)a$. On the other hand, it is the norm of the multihomogenization of α, which is a multihomogeneous polynomial in the $W_j^{(i)}$, with respect to the field extension

$$\tilde{L}/\bar{\mathbb{Q}}\left(V_j^{(i)}; 1 \le i \le m, 0 \le j \le u_i\right),$$

where $\tilde{L}$ is the fraction field of the multihomogeneous coordinate ring of $Y_1 \times \cdots \times Y_m \hookrightarrow \mathbb{P}^{N_1} \times \cdots \times \mathbb{P}^{N_m}$. As such G is integral over R and therefore lies in R. So G is in fact a multihomogeneous polynomial of the desired multidegree.

Note that $\beta = N(\alpha)\alpha^{-1} = \prod_{\tau \neq \mathrm{id}} \tau(\alpha)$ lies in $\bar{\mathbb{Q}}(\mathcal{Y})$ and is integral over the local ring $\mathcal{O}_{\mathbb{P}^{u_1} \times \cdots \times \mathbb{P}^{u_m}, \rho(x)}$ because α is. Here, τ runs over the embeddings of $\bar{\mathbb{Q}}(\mathcal{Y})$ into a normal closure of the extension $\bar{\mathbb{Q}}(\mathcal{Y})/L$. Hence, β is integral over $\mathcal{O}_{\mathcal{Y},x}$. As ρ is étale at x, this local ring is normal and hence contains β. So the index of $N(\alpha)$ in x (or equivalently, the index of G in $\rho(x)$) is greater than or equal to the index of α in x. For the bound for $h(G)$, see Lemme 5.5 in [154].

We can then apply Théorème 5.6 (Faltings' product theorem) from [154] with the value of σ_0 above and obtain in this way a contradiction with Proposition A.2.1. The hypotheses of the theorem are satisfied since

$$\frac{a_i}{a_{i+1}} \ge c_2 \ge \left(\frac{m}{\sigma_0}\right)^u \ge (2u^2)^{u^2}$$

and G has index at least σ with respect to the weight $dt_1 a$ in $\rho(x)$, hence has index at least $\sigma \prod_{i=1}^m d_i^{-1} \ge \sigma_0$ with respect to the weight $dt_1(d_1 \cdots d_m)a$ in $\rho(x)$.

We obtain a pair (l, U) with $U\left(V^{(l)}\right)(x_l) = 0$, U non-zero, $\deg(U) \le \left(\frac{m}{\sigma_0}\right)^u = \Lambda^{\omega' u \psi(u)}$, and

$$\begin{aligned} a_l h(U) \le{}& u_l \left(\frac{m}{\sigma_0}\right)^u \left(\frac{h(G)}{dt_1 d_1 \cdots d_m} + \sum_{i=1}^m a_i(u_i \log(u_i+1) + \log 2)\right) \\ &+ a_l \left(\frac{m}{\sigma_0}\right)^u (u_l+1) \log \left(\frac{m}{\sigma_0}\right)^u (u_l+1) + a_l \log \binom{\deg(U) + u_l}{u_l} + o(1). \end{aligned}$$

Here, we used that the height of projective n-space is bounded from above by $n \log(n+1)$.

After some simplification and by using that $u_l \ge 1$ (which is a consequence of the product theorem) and $\frac{m}{\sigma_0} = \Lambda^{\omega' \psi(u)}$, we deduce that

$$\begin{aligned} a_l h(U) &\le u_l \Lambda^{\omega' u \psi(u)} \left(\frac{h(s_d)}{dt_1} + 2\Lambda_h + 2a_l \log 2 \left(\frac{m}{\sigma_0}\right)^u (u_l+1)\right) + o(1) \\ &\le u_l \Lambda^{\omega' u \psi(u)} \left(\frac{2M'}{u\epsilon}(1 + 2t_2 + 3\epsilon)\Lambda_h + 2a_l \log \left(\frac{m}{\sigma_0}\right)^u\right) + o(1). \end{aligned}$$

For the last inequality, we used that $2a_l \log 2(u_l+1) \le 2\Lambda_h$ and $\frac{2M'}{u} \ge 2$.

We can now estimate

$$2a_l u_l \log \left(\frac{m}{\sigma_0}\right)^u \le 2\Lambda_h \omega' u \psi(u) \log \Lambda \le \Lambda_h \Lambda^{(\omega' u - 1)\psi(u)}$$

since $a_l u_l \le \Lambda_h$, $2\omega' u\psi(u)\log\Lambda \le \Lambda^{\frac{1}{2}\omega' u\psi(u)}$, and $\frac{1}{2}\omega' u \le \omega' u - 1$. Thanks to (iii), we can bound the first term as

$$\begin{aligned}\frac{2M'u_l}{u\epsilon} \le \frac{2M'}{\epsilon} \le 2M\max_i (N_i+1)^m(2u\theta)(t_1 m)^u \prod_{i=1}^m \deg(X_i)^{1+\omega}\Lambda^{(1+\omega)(\psi(u)-1)}\\ \le M\Lambda^m\Lambda^{(1+\omega)\psi(u)} \le M\Lambda^{(\omega' u-1)\psi(u)}\end{aligned}$$

since $m + (2+\omega)\psi(u) \le \omega' u\psi(u)$. This last inequality follows from $m \le u\psi(u)$.

Combining these inequalities with the one above, we obtain that

$$a_l h(U) \le \Lambda^{(2\omega' u-1)\psi(u)} M\Lambda_h(2+2t_2+3\epsilon) + o(1) \le \Lambda^{2\psi(u-1)-2\psi(u)}\left(\frac{Mt_2}{4d_l}\right)\Lambda_h,$$

where we used that $2+2t_2+3\epsilon \le 5t_2 \le \frac{\Lambda^{\psi(u)}t_2}{4d_l}$ by (ii). We could get rid of the $o(1)$ since, for example, this last inequality is in fact strict. Thus, we have found a contradiction with Proposition A.2.1. □

A.5. Finishing the proof

We now have established that the section $s_d \in \Gamma(\mathcal{Y}, \mathcal{Q}_d)$ given by Proposition A.3.1 has index (in x and with respect to the weight $dt_1 a$) bounded as

$$\sigma \le (4t_1 \max_i d_i(N_i+1))^{-1}\epsilon.$$

We write D for a differential operator associated to that index and finish the proof of Theorem A.1.1 by considering the following height

$$\begin{aligned}-h_{\mathcal{Q}_d}(x) = dh(Z(x)) - dh(\Xi(x)) + d\epsilon\sum_{i=1}^m a_i h(W^{(i)}(x))\\ = h(Z'_d(x)) - dh(\Xi(x)).\end{aligned}$$

By definition, there exists such a D with $D(s_d)(x) \ne 0$ and we have $D'(s_d)(x) = 0$ for every operator D' of index $\sigma' < \sigma$, hence by the product formula,

$$h(Z'_d(x)) = h\left((D(s_d)\otimes Z'_d)(x)\right) = h\left((D(s_d\otimes\zeta')(x))_{\zeta'\in Z'_d}\right).$$

In order to define the right-hand side, one has to fix an isomorphism $\mathcal{P}^{\otimes d} \simeq \mathcal{O}_{\mathcal{Y}}$. The right-hand side is however independent of the choice of isomorphism, precisely since D is an operator associated to the index of s_d.

Let us recall that the sections $s_d \otimes \zeta' \in \Gamma(\mathcal{Y}, \mathcal{P}^{\otimes d})$ are homogeneous polynomials of degree d in the sections Ξ and that $\log\dim\Gamma(\mathcal{Y}, \mathcal{P}^{\otimes d}) = o(d)$. Furthermore, the sections Ξ themselves are monomials of multidegree $t_1 a$ in the coordinates $W^{(i)}$. Hence, the right choice of isomorphism shows that

$$h(Z'_d(x)) \le h(s_d) + h\left((D(\xi^\nu)(x))_{\xi^\nu}\right) + o(d),$$

where ξ^ν runs over the monomials of degree d in the sections Ξ (seen as monomials of multidegree $dt_1 a$ in the $W^{(i)}$) divided by appropriate products of the $V_0^{(i)}$ ($i = 1, \ldots, m$).

We can estimate the height of the $D(\xi^\nu)(x)$ by using Leibniz' formula as well as Corollaire 5.1 and Lemme 5.2 from [154] (corrected). For $1 \le i \le m$ and $l = (l_1, \dots, l_{u_i}) \in (\mathbb{N} \cup \{0\})^{u_i}$, we define the operator

$$\partial^{i,l} = \prod_{j=1}^{u_i} \frac{1}{l_j!} \left(\partial_{V_j^{(i)}/V_0^{(i)}} \right)^{l_j} : \mathcal{O}_{\mathcal{Y},x} \to \mathcal{O}_{\mathcal{Y},x}.$$

If $w = (w_1, \dots, w_k) \in (\mathbb{N} \cup \{0\})^k$ is a multi-index, we write $|w| = w_1 + \cdots + w_k$.

LEMMA A.5.1. *Let $1 \le i \le m$ be an integer and let K be a number field that contains the coordinates $\left(W_j^{(i)}/V_0^{(i)}\right)(x)$, the families B_i, and the products*

$$c_i = \prod_j \left(\frac{\left(W_j^{(i)}/V_0^{(i)}\right)^{b_{i,j}}}{b_{i,j}!} \frac{\partial^{b_{i,j}} Q_j^{(i)}}{\partial T_{u_i+1}^{b_{i,j}}} \left(1, \frac{V_1^{(i)}}{V_0^{(i)}}, \dots, \frac{V_{u_i}^{(i)}}{V_0^{(i)}}, \frac{W_j^{(i)}}{V_0^{(i)}}\right)(x_i) \right)^{\frac{1}{b_{i,j}}},$$

where j runs over the indices satisfying $W_j^{(i)} \neq 0$ in $\Gamma(Y_i, \mathcal{L}_i)$. Then for every place v of K and every multi-index $l \in (\mathbb{N} \cup \{0\})^{u_i}$, we have

$$\left| \partial^{i,l} \left(W_j^{(i)}/V_0^{(i)}\right)(x) \right|_v \le \left| \left(W_j^{(i)}/V_0^{(i)}\right)(x) \right|_v (|c_i|_v^{-2} C_{i,v})^{|l|}$$

with

$$C_{i,v} = 2^{-\epsilon_v} \left((d_i(N_i+1))^{6d_i\epsilon_v} \max_{b \in B_i} |b|_v \max_{0 \le k \le N_i} \left| \left(W_k^{(i)}/V_0^{(i)}\right)(x) \right|_v^{d_i} \right)^{2(N_i+1)}$$

and $\epsilon_v = 1$ if v is infinite, 0 if v is finite.

PROOF. Recall that by Corollary A.2.2, the number $c_i \in \bar{\mathbb{Q}} \backslash \{0\}$ is well defined (up to the choice of the roots which can be made arbitrarily).

If $W_j^{(i)} = 0$ in $\Gamma(Y_i, \mathcal{L}_i)$, the derivative $\partial^{i,l}\left(W_j^{(i)}/V_0^{(i)}\right)$ is zero and the inequality holds. Otherwise, we may apply Corollaire 5.1 from [154] and follow the proof of Lemme 5.2 in [154] with $N = N_i$, using at the end that

$$2f_2(u_i, d_i) = 2(2u_i+4)^{1+\frac{3}{2}d_i} d_i \binom{d_i+u_i}{u_i}^{2(N_i+2)} \binom{d_i+1}{2}^2 (2(N_i+1)d_i)^{2(N_i+1)d_i}$$

is bounded from above by

$$\begin{aligned} &2^{2+\frac{3}{2}d_i+2(N_i+1)d_i}(N_i+2)^{1+\frac{3}{2}d_i}(N_i+1)^{2(N_i+2)d_i} d_i^5 ((N_i+1)d_i)^{2(N_i+1)d_i} \\ &\le (N_i+1)^{N_i+1+\frac{3}{2}d_i+\frac{\log 3}{\log 2}(1+\frac{3}{2}d_i)+5(N_i+1)d_i} d_i^5 ((N_i+1)d_i)^{2(N_i+1)d_i} \\ &\le (N_i+1)^{2(N_i+1)+2(N_i+1)d_i+5(N_i+1)d_i} d_i^5 ((N_i+1)d_i)^{2(N_i+1)d_i} \\ &\le (d_i(N_i+1))^{12d_i(N_i+1)}. \end{aligned}$$

□

For $D = \prod_{i=1}^m \partial^{i,\kappa_i}$, we obtain the following bound (cf. the proof of Proposition 5.3 in [154]):

$$|D(\xi^\nu)(x)|_v \le |\xi^\nu(x)|_v \prod_{i=1}^m 2^{(|\kappa_i|+dt_1u_ia_i)\epsilon_v} (|c_i|_v^{-2} C_{i,v})^{|\kappa_i|}$$

and hence thanks to the product formula for the c_i

$$h\left((D(\xi^\nu)(x))_{\xi^\nu}\right) \leq dh(\Xi(x)) + \sum_{i=1}^{m} 2(N_i+1)|\kappa_i|(h(B_i) + d_i h(W^{(i)}(x))) \\ + \sum_{i=1}^{m} (dt_1 u_i a_i \log 2 + 12 d_i (N_i+1)|\kappa_i| \log d_i(N_i+1)).$$

We proceed with bounding

$$|\kappa_i| \leq dt_1 a_i \sigma \leq (4(N_i+1)d_i)^{-1} d\epsilon a_i,$$

which implies that $\sum_{i=1}^{m} 2d_i(N_i+1)|\kappa_i| h(W^{(i)}(x)) \leq \frac{d\epsilon}{2} h_{\mathcal{N}_a}(x)$. Together with (A.2.3), the bound also implies that

$$\sum_{i=1}^{m} 2(N_i+1)|\kappa_i|(h(B_i) + 6d_i \log d_i(N_i+1)) \\ \leq d\epsilon \sum_{i=1}^{m} a_i \left(\frac{h_i + d_i(u_i+1)\log d_i(N_i+1)}{2d_i} + 3\log d_i(N_i+1) \right) \leq 4d\epsilon \Lambda_h.$$

Finally, we know that $\sum_{i=1}^{m} u_i a_i \log 2 \leq \Lambda_h$ and putting all these estimates together, we get

$$\epsilon h_{\mathcal{N}_a}(x) - h_{\mathcal{M}}(x) = -\frac{h_{\mathcal{Q}_d}(x)}{d} \leq \frac{h(s_d)}{d} + \frac{\epsilon}{2} h_{\mathcal{N}_a}(x) + (t_1 + 4\epsilon)\Lambda_h + o(1).$$

Thanks to Proposition A.3.1 and (A.2.1), it follows that

$$\frac{\epsilon}{2} h_{\mathcal{N}_a}(x) - h_{\mathcal{M}}(x) \leq \frac{2M'}{u\epsilon}(t_1 + 2t_2 + \epsilon)\Lambda_h + (t_1 + 4\epsilon)\Lambda_h + o(1) \\ \leq \left(\frac{2M'}{u\epsilon}\right)(2t_1 + 2t_2 + 5\epsilon)\Lambda_h + o(1) \\ \leq \left(\frac{M' t_1}{\epsilon^2}\right) 8(2 + 2t_2 + 5\epsilon)\Lambda^{2\psi(u) - 2\psi(0)} (Mt_2)^{-u} \left(\frac{\epsilon}{4}\sum_{i=1}^{m} a_i c_3^{(i)}\right),$$

where the strict inequality in (A.2.1) allowed us to sweep the $o(1)$ under the rug (for d large enough).

We have $8(2 + 2t_2 + 5\epsilon) \leq 42 t_2 \leq \Lambda^2 t_2$ and it follows from (iii) that

$$(M' t_1)\epsilon^{-2} \leq M t_1 \max_i (N_i+1)^m (2u\theta)^2 (t_1 m)^{2u} \left(\prod_{i=1}^{m} \deg(X_i)\right)^{2(1+\omega)} \Lambda^{2(1+\omega)(\psi(u)-1)} \\ \leq M\Lambda^{\max\{3,m\} + 2(1+\omega)\psi(u)} \leq M\Lambda^{2\omega' u\psi(u) - 2} = M\Lambda^{2\psi(u-1) - 2\psi(u) - 2},$$

where we used that $\max\{3, m\} \leq 4u\psi(u) - 2$.

Hence, we can deduce that

$$\frac{\epsilon}{2} h_{\mathcal{N}_a}(x) - h_{\mathcal{M}}(x) \leq (Mt_2)^{-(u-1)} \Lambda^{2\psi(u-1) - 2\psi(0)} \frac{\epsilon}{4} \sum_{i=1}^{m} a_i c_3^{(i)} \leq \frac{\epsilon}{4} h_{\mathcal{N}_a}(x),$$

from which it follows that $h_{\mathcal{N}_a}(x) \leq 4\epsilon^{-1} h_{\mathcal{M}}(x)$. The theorem follows since by (iii)

$$4\epsilon^{-1} \leq 8u\theta(t_1 m)^u \left(\prod_{i=1}^{m} \deg(X_i)\right)^{1+\omega} \Lambda^{(1+\omega)(\psi(u)-1)} \leq \Lambda^{(1+\omega)\psi(u)+2}$$

and $\Lambda^{(1+\omega)\psi(u)+2} \leq \Lambda^{\omega' u \psi(u)} \leq c_1$.

APPENDIX B

On the frequency of height values

Wer A sagt, der muß nicht B sagen.
Er kann auch erkennen, daß A
falsch war.

B. Brecht, *Der Jasager*

B.1. Introduction

Fix an embedding of $\bar{\mathbb{Q}}$ into $\mathbb{C}$. Recall that $H(\alpha)$ denotes the absolute multiplicative Weil height of $\alpha \in \bar{\mathbb{Q}}$. For $d \in \mathbb{N}$, $k \in \{0, \ldots, d\}$, and $\mathcal{H} \in [1, \infty)$, we set

$$A(k,d,\mathcal{H}) = \{\alpha \in \mathbb{C}; [\mathbb{Q}(\alpha):\mathbb{Q}] = d, H(\alpha) = \mathcal{H}, \text{ and precisely } k \text{ conjugates of } \alpha \text{ lie inside the open unit disk}\},$$

$$A(k,d) = \bigcup_{\mathcal{H} \geq 1} A(k,d,\mathcal{H}),$$

$$B(k,d,\mathcal{H}) = \{H(\alpha); \alpha \in \mathbb{C}, [\mathbb{Q}(\alpha):\mathbb{Q}] = d, H(\alpha) \leq \mathcal{H}, \text{ and precisely } k \text{ conjugates of } \alpha \text{ lie inside the open unit disk}\},$$

and

$$B(k,d) = \{H(\alpha); \alpha \in A(k,d)\}.$$

The goal of this chapter is to measure the growth of these sets in terms of $\mathcal{H}$.

We set

$$a(k,d) = \lim_{\substack{\mathcal{H} \in B(k,d) \\ \mathcal{H} \to \infty}} \frac{\log |A(k,d,\mathcal{H})|}{\log \mathcal{H}}$$

and

$$b(k,d) = \lim_{\mathcal{H} \to \infty} \frac{\log |B(k,d,\mathcal{H})|}{\log \mathcal{H}}$$

if these limits exist.

We remark that it is not clear if the conjugates inside the open unit disk are the right thing to take into account here. The Galois group of the normal closure of $\mathbb{Q}(\alpha)$ and the degree $[\mathbb{Q}\left(H(\alpha)^d\right) : \mathbb{Q}]$ also seem to play an important role as will become apparent. Of course, these objects are not independent of one another (e.g. $k \in \{0, d\}$ is equivalent to $[\mathbb{Q}\left(H(\alpha)^d\right) : \mathbb{Q}] = 1$).

Much is known about counting algebraic numbers or more generally points in $\mathbb{P}^n(\bar{\mathbb{Q}})$ of fixed degree (over $\mathbb{Q}$ or over any fixed number field) and bounded height: Schanuel first proved, in [162], an asymptotic for the number of algebraic points of bounded height that are defined over a fixed number field. Further results, including the asymptotic for the number of quadratic points (over $\mathbb{Q}$) of bounded height, were obtained by Schmidt in [163] and [164]. If n is larger than the degree of the point (over $\mathbb{Q}$), then Gao found and proved the correct asymptotic in [49]. Masser and Vaaler then counted algebraic numbers of fixed degree and bounded height in [98] (over $\mathbb{Q}$) and [97] (over any fixed number field).

If the degree of the point (over any fixed number field) is slightly less than $\frac{2n}{5}$, then Widmer obtained the correct asymptotic in [190]. In [191], Widmer counted integral algebraic points of fixed degree (over any fixed number field) and bounded height under the assumption that the degree of the point is either 1 or slightly less than n. In [9], Barroero counted algebraic integers of fixed degree (over any fixed number field) and bounded height. In [60], Grizzard and Gunther counted (among other things) algebraic integers of fixed degree (over $\mathbb{Q}$), fixed norm and bounded height. This last result is somewhat related to our work in that the d-th power of the height of an algebraic integer of degree d with no conjugate inside the open unit disk is equal to the absolute value of its norm.

We emphasize that all these results give much more precise asymptotics than the ones obtained in this chapter. However, already when counting rational numbers of fixed height, Euler's phi function appears, so it is clear that such precise asymptotics cannot be obtained in general when counting algebraic numbers of fixed degree and fixed height.

The main results of this chapter can be summarized as follows:

- $a(0,d) = a(d,d) = d^2$ (Theorem B.2.1);
- $b(0,d) = b(d,d) = d$ (Theorem B.2.1);
- $b(k,d) = d(d+1)$ if $0 < k < d$ (Theorem B.4.1(ii));
- $a(k,d) = 0$ if $0 < k < d$ and $\gcd(k,d) = 1$ (Theorem B.5.1);
- $a(k,d)$ does not exist if $0 < k < d$ and $\gcd(k,d) > 1$, but the corresponding limes inferior and limes superior are equal to 0 and $d(\gcd(k,d) - 1)$ respectively ((B.8.3) and its proof and Theorem B.6.2).

Furthermore, if $d = 4$ and $k = 2$, then we obtain finer results according to whether $[\mathbb{Q}(\mathcal{H}^4) : \mathbb{Q}]$ equals 2, 4, or 6. In the last two cases, we will see that $|A(2,4,\mathcal{H})|$ grows more slowly than $\mathcal{H}^\epsilon$ for every positive ϵ (Lemma B.3.3 and Theorem B.7.1). In the first case, we will find that any limit κ between 0 and 4 can be achieved along a suitable subsequence of height values (Theorem B.6.1). In the construction in the proof of Theorem B.6.1, the field $\mathbb{Q}(\mathcal{H}^4)$ is made to vary in an infinite set (unless $\kappa = 4$).

In Section B.8, we count polynomials with integer coefficients of fixed degree and fixed Mahler measure. Following a suggestion of Norbert A'Campo, we study the dynamical behaviour of the height function in Section B.9. The dynamical behaviour of the Mahler measure has been studied initially by Dubickas in [40] and [41] and subsequently by Zhang in [194] as well as by Fili, Pottmeyer, and Zhang in [46].

Our proofs are mostly elementary. Our constructions of many algebraic numbers of a given height rely on point counting results for lattices by Barroero-Widmer in [14] (generalizing a theorem of Davenport in [35]) and Technau-Widmer in [177].

For a real number ξ, we denote by $[\xi]$ the largest integer which does not exceed ξ. We use ϕ to denote Euler's phi function and μ to denote the Möbius function. The following simple observation will be used at different places throughout this chapter: If α is an algebraic number of degree d, a is the leading coefficient of a minimal polynomial of α in $\mathbb{Z}[t]$ (there are two choices for a minimal polynomial of α in $\mathbb{Z}[t]$ as $(\mathbb{Z}[t])^* = \{\pm 1\}$), and $\alpha_1, \ldots, \alpha_k$ are the conjugates of α that lie outside the open unit disk, then $H(\alpha)^d = |a||\alpha_1| \cdots |\alpha_k| = \pm a\alpha_1 \cdots \alpha_k$. We can write $\pm\alpha_i$ instead of $|\alpha_i|$ $(i = 1, \ldots, k)$ since the non-real conjugates appear in complex conjugate pairs and the real conjugates are equal to their absolute value up to sign.

B.2. The case $k \in \{0, d\}$

THEOREM B.2.1. *Let $d \in \mathbb{N}$. The following hold:*

(i) $b(0, d) = b(d, d) = d$;

(ii) $a(0, d) = a(d, d) = d^2$.

(In particular, all these limits exist.)

PROOF. (i) Eisenstein's criterion shows that all real positive d-th roots of integers between 2 and $\mathcal{H}^d$ that are congruent to 2 modulo 4 belong to $B(0, d, \mathcal{H})$. Using that the height of a non-zero algebraic number is equal to the height of its inverse, we deduce that they also belong to $B(d, d, \mathcal{H})$. Also, every element of $B(0, d, \mathcal{H})$ or $B(d, d, \mathcal{H})$ is a real positive d-th root of some integer between 1 and $\mathcal{H}^d$. So $\mathcal{H}^d \geq |B(0, d, \mathcal{H})| \geq \frac{1}{5}\mathcal{H}^d$ for $\mathcal{H}$ large enough and the same holds for $|B(d, d, \mathcal{H})|$.

(ii) Let us define

$$\begin{aligned} Z = \Big\{ &(w_0, \ldots, w_{d-1}, T) \in \mathbb{R}^d \times \mathbb{R}; w_0 > 0, \exists x_1, \ldots, x_d,\ y_1, \ldots, y_d \in \mathbb{R} : \\ &x_j^2 + y_j^2 \geq 1\ \forall j = 1, \ldots, d,\ g_j(x_1, y_1, \ldots, x_d, y_d) = 0\ \forall j = 0, \ldots, d-1, \\ &\text{and } f_j(x_1, y_1, \ldots, x_d, y_d) = w_j \forall j = 0, \ldots, d-1 \Big\}, \end{aligned} \tag{B.2.1}$$

where $f_j(x_1, y_1, \ldots, x_d, y_d) =$

$$\mathrm{Re}\left(\frac{(-1)^{d-j}T}{(x_1 + \sqrt{-1}y_1) \cdots (x_d + \sqrt{-1}y_d)} \sigma_j(x_1 + \sqrt{-1}y_1, \ldots, x_d + \sqrt{-1}y_d) \right),$$

and $g_j(x_1, y_1, \ldots, x_d, y_d) =$

$$\mathrm{Im}\left(\frac{1}{(x_1 + \sqrt{-1}y_1) \cdots (x_d + \sqrt{-1}y_d)} \sigma_j(x_1 + \sqrt{-1}y_1, \ldots, x_d + \sqrt{-1}y_d) \right)$$

for $j = 0, \ldots, d-1$. Here, σ_j is the j-th elementary symmetric polynomial in d variables, $\sqrt{-1}$ denotes the imaginary unit in $\mathbb{C}$, and Re and Im denote the real and the imaginary part of a complex number respectively.

The set Z is definable in the o-minimal structure of all semialgebraic subsets of $\mathbb{R}^n$ $(n \in \mathbb{N})$; see Section 3.5 for an introduction to o-minimal structures. Let $\pi : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d$ be the canonical projection. For $T \in \mathbb{R}$, $T \neq 0$, the set $Z_T = \pi(Z \cap (\mathbb{R}^d \times \{T\}))$ parametrizes polynomials of degree d with real coefficients and positive leading coefficient that have no complex zeroes inside the open unit disk and whose constant coefficient is equal to T. Note that $Z_T = |T| \cdot Z_{T/|T|}$ $(T \neq 0)$ and that the coordinates of a point in Z_T can all be bounded by some constant multiple of $|T|$, depending on d. It follows that the volume of Z_T is $|T|^d$ times the volume of $Z_{T/|T|}$ $(T \neq 0)$ and that the volume of any orthogonal projection of Z_T on some

j-dimensional coordinate subspace of $\mathbb{R}^d$ ($j \leq d-1$) has volume at most a constant multiple of $|T|^{d-1}$.

It then follows from Theorem 1.3 in [14] that

$$\left|\left|Z_T \cap \mathbb{Z}^d\right| - V_{T/|T|}|T|^d\right| = \mathcal{O}(|T|^{d-1})$$

for $|T| \geq 1$, where V_u is the volume of Z_u for $u \in \{\pm 1\}$ (positive since Z_u has non-empty interior). We have

$$\begin{aligned} N_d(T) := |\{P(t) = at^d + \cdots \pm T \in \mathbb{Z}[t]; a > 0, \text{ all zeroes of } P \text{ are at least} \\ 1 \text{ in absolute value}\}| = \left|Z_T \cap \mathbb{Z}^d\right| + \left|Z_{(-T)} \cap \mathbb{Z}^d\right| = (V_1 + V_{-1})T^d + \mathcal{O}(T^{d-1}) \end{aligned} \tag{B.2.2}$$

for $T \in \mathbb{N}$.

If we define

$$\begin{aligned} \tilde{N}_d(T) = |\{P(t) = at^d + \cdots \pm T \in \mathbb{Z}[t]; a > 0, \gcd(a, \ldots, \pm T) = 1, \\ \text{all zeroes of } P \text{ are at least 1 in absolute value}\}|, \end{aligned} \tag{B.2.3}$$

then we have $N_d(T) = \sum_{S|T} \tilde{N}_d(S)$. Using Möbius inversion together with an elementary bound for the divisor function, we deduce that

$$\tilde{N}_d(T) = \sum_{S|T} \mu(S) N_d\left(\frac{T}{S}\right) = (V_1 + V_{-1})T^d \left(\sum_{S|T} \frac{\mu(S)}{S^d}\right) + \mathcal{O}\left(T^{d-\frac{1}{2}}\right).$$

Here $\sum_{S|T} \frac{\mu(S)}{S^d} = \prod_{p|T}\left(1 - \frac{1}{p^d}\right)$ is at most 1 and at least $\frac{\phi(T)}{T}$, so it decays more slowly than $T^{-\epsilon}$ for any $\epsilon > 0$. In fact, for $d \geq 2$, the product is at least $\prod_{k=2}^{\infty}\left(1 - \frac{1}{k^2}\right) = \frac{1}{2}$, so bounded from below uniformly.

What we really want is

$$\begin{aligned} \hat{N}_d(T) = |\{P(t) = at^d + \cdots \pm T \in \mathbb{Z}[t]; a > 0, \gcd(a, \ldots, \pm T) = 1, \\ P \text{ is irreducible, and all zeroes of } P \text{ are at least 1 in absolute value}\}| \end{aligned} \tag{B.2.4}$$

since $|A(0, d, \mathcal{H})| = d\hat{N}_d(\mathcal{H}^d)$ if $\mathcal{H}^d \in \mathbb{N}$, but the contribution of the reducible polynomials to $\tilde{N}_d(T)$ is at most

$$\sum_{e=1}^{[\frac{d}{2}]} \sum_{R|T} \tilde{N}_e(R)\tilde{N}_{d-e}\left(\frac{T}{R}\right) = \sum_{e=1}^{[\frac{d}{2}]} \sum_{R|T} \mathcal{O}(R^{2e-d}T^{d-e}) = \mathcal{O}\left(T^{d-\frac{1}{2}}\right).$$

Hence, we obtain that

$$d(V_1+V_{-1})\frac{\phi(\mathcal{H}^d)}{\mathcal{H}^d}\mathcal{H}^{d^2} - \mathcal{O}\left(\mathcal{H}^{d\left(d-\frac{1}{2}\right)}\right) \leq |A(0, d, \mathcal{H})| \leq d(V_1+V_{-1})\mathcal{H}^{d^2} + \mathcal{O}\left(\mathcal{H}^{d\left(d-\frac{1}{2}\right)}\right)$$

for $\mathcal{H} \in B(0, d)$ and (ii) follows, at least for $a(0, d)$. For $a(d, d)$ we can repeat the same argument, but counting $\frac{1}{\alpha}$ instead of α and replacing $x_j^2 + y_j^2 \geq 1$ by $x_j^2 + y_j^2 > 1$ in (B.2.1). $\square$

One can say even more about the sets $B(0, d)$ and $B(d, d)$. We denote by $\mathbb{N}^{\frac{1}{d}}$ the set of the positive real d-th roots of all natural numbers.

LEMMA B.2.2. *Let $d \in \mathbb{N}$. We have*

$$B(0,d) = \begin{cases} \mathbb{N}^{\frac{1}{d}} \backslash \{1\} & \text{if } d \notin \phi(\mathbb{N}), \\ \mathbb{N}^{\frac{1}{d}} & \text{if } d \in \phi(\mathbb{N}), \end{cases}$$

and

$$B(d,d) = \begin{cases} \mathbb{N}^{\frac{1}{d}} \backslash \{1\} & \text{if } d > 1, \\ \mathbb{N}^{\frac{1}{d}} & \text{if } d = 1. \end{cases}$$

PROOF. (As suggested by G. Rémond.) It follows from the definition of the height that $B(0,d)$ and $B(d,d)$ are both contained in $\mathbb{N}^{\frac{1}{d}}$. In the case $d = 1$, the lemma follows from $H(n) = H(n^{-1}) = n$ for all $n \in \mathbb{N}$ together with $H(0) = 1$, so we assume that $d \geq 2$.

If $[\mathbb{Q}(\alpha) : \mathbb{Q}] = d \geq 2$ and $H(\alpha) = 1$ for some $\alpha \in \bar{\mathbb{Q}}$, then α is a root of unity by Kronecker's theorem, so $\alpha \in A(0,d,1)$ and $d = \phi(n)$ for some $n \in \mathbb{N}$. On the other hand, if $d = \phi(n)$ for some $n \in \mathbb{N}$, then any primitive n-th root of unity belongs to $A(0,d,1)$. It follows that 1 never belongs to $B(d,d)$ and that 1 belongs to $B(0,d)$ if and only if $d \in \phi(\mathbb{N})$.

Let now N be a natural number that is greater than or equal to 2. We want to show that the positive real d-th root $N^{\frac{1}{d}}$ of N belongs to $B(0,d) \cap B(d,d)$. For this, we define a natural number p as follows: If $N = 2$, we set $p = 1$. If $N \geq 3$, then we let $p \in \mathbb{N}$ be a prime number such that $p < N$ and p does not divide N. Such a prime number always exists: If $N = 3$, we set $p = 2$. If $N \geq 4$ and no such prime number existed, then N would be divisible by the product Π of all prime numbers that are smaller than N. Now $\Pi - 1 \geq 5$ must have a prime factor and this prime factor must be greater than or equal to N. It follows that $\Pi \leq N \leq \Pi - 1$, a contradiction.

The polynomial $Nt^d - p$ is irreducible in $\mathbb{Z}[t]$ by the coprimality of p and N together with Eisenstein's criterion (applied to $pt^d - N$ if $N = 2$). Its zeroes belong to $A\left(d,d,N^{\frac{1}{d}}\right)$ and their inverses belong to $A\left(0,d,N^{\frac{1}{d}}\right)$. It follows that $N^{\frac{1}{d}} \in B(0,d) \cap B(d,d)$. This proves the lemma. $\square$

B.3. Some useful lemmata

LEMMA B.3.1. *Let $d \in \mathbb{N}$ and $k \in \{0,\dots,d\}$. The limit*

$$\lim_{\mathcal{H} \to \infty} \frac{\sum_{\mathcal{H}' \leq \mathcal{H}} |A(k,d,\mathcal{H}')|}{\mathcal{H}^{d(d+1)}}$$

exists and is positive.

PROOF. We first remark that

$$\sum_{\mathcal{H}' \leq \mathcal{H}} |A(k,d,\mathcal{H}')| = |\{\alpha \in \mathbb{C}; [\mathbb{Q}(\alpha) : \mathbb{Q}] = d, H(\alpha) \leq \mathcal{H}, \text{ and}$$
$$\text{precisely } k \text{ conjugates of } \alpha \text{ lie inside the open unit disk}\}|.$$

We can again apply Theorem 1.3 from [14] to the following definable family of semialgebraic sets:

$$\tilde{Z} = \Bigg\{(w_0, \ldots, w_d, T) \in \mathbb{R}^{d+1} \times \mathbb{R}; T \geq 1, w_0 > 0, \exists x_1, \ldots, x_d,$$
$$y_1, \ldots, y_d \in \mathbb{R} : x_j^2 + y_j^2 < 1 \ \forall j = 1, \ldots, k,\ x_j^2 + y_j^2 \geq 1 \ \forall j = k+1, \ldots, d,$$
$$g_j(x_1, y_1, \ldots, x_d, y_d) = 0 \ \forall j = 1, \ldots, d,$$
$$w_0 f_j(x_1, y_1, \ldots, x_d, y_d) = w_j \forall j = 1, \ldots, d, w_0^2 \prod_{j=k+1}^{d} (x_j^2 + y_j^2) \leq T^2 \Bigg\}, \quad \text{(B.3.1)}$$

where

$$f_j(x_1, y_1, \ldots, x_d, y_d) = (-1)^j \operatorname{Re}\left(\sigma_j(x_1 + \sqrt{-1}y_1, \ldots, x_d + \sqrt{-1}y_d)\right)$$

and

$$g_j(x_1, y_1, \ldots, x_d, y_d) = \operatorname{Im}\left(\sigma_j(x_1 + \sqrt{-1}y_1, \ldots, x_d + \sqrt{-1}y_d)\right)$$

for $j = 1, \ldots, d$ and the σ_j are again the elementary symmetric polynomials in d variables. If again $\tilde{Z}_T = \pi(\tilde{Z} \cap (\mathbb{R}^{d+1} \times \{T\}))$ for the projection $\pi : \mathbb{R}^{d+1} \times \mathbb{R} \to \mathbb{R}^{d+1}$ and $T \geq 1$, then it is easy to see that all coordinates of a point in $\tilde{Z}_T$ are bounded by some constant multiple of T, depending on d, that $\tilde{Z}_T = T \cdot \tilde{Z}_1$, and that $\tilde{Z}_1$ has non-empty interior.

Similarly as above, $N_{d,k}(T) := |\tilde{Z}_T \cap \mathbb{Z}^{d+1}|$ counts the number of polynomials $P(t) \in \mathbb{Z}[t]$ of degree d with positive leading coefficient and precisely k zeroes inside the open unit disk such that the product of the leading coefficient and the absolute values of the zeroes outside the open unit disk is at most T. If $\tilde{N}_{d,k}(T)$ denotes the number of such polynomials with coprime coefficients, then we have that $N_{d,k}(T) = \sum_{n=1}^{\infty} \tilde{N}_{d,k}\left(\frac{T}{n}\right)$.

Using another Möbius inversion and Theorem 1.3 from [14], we deduce that

$$\tilde{N}_{d,k}(T) = C\left(\sum_{n=1}^{\infty} \frac{\mu(n)}{n^{d+1}}\right) T^{d+1} + \mathcal{O}(T^d \log T)$$

for some constant $C > 0$. The proof of Lemma 2 in [98] shows that the number of reducible polynomials that we count in this way is of lower growth order. We can therefore deduce the lemma by setting $T = \mathcal{H}^d$. □

LEMMA B.3.2. *Let $d \in \mathbb{N}$ and $k \in \{0, \ldots, d\}$. If the limits $a(k,d)$ and $b(k,d)$ both exist, then $a(k,d) + b(k,d) = d(d+1)$.*

PROOF. If they added up to some smaller number, we immediately obtain a contradiction with Lemma B.3.1 for $\mathcal{H}$ big enough, so suppose they add up to some bigger number. If $b(k,d) = 0$, then $a(k,d) > d(d+1)$ and we immediately get a contradiction with Lemma B.3.1 for $\mathcal{H}$ big enough. So we can assume that $b(k,d) > 0$.

We can find some $\epsilon \in (0,1)$ such that $(1-\epsilon)b(k,d) + (1-\epsilon)a(k,d) > d(d+1)$ and then we can find $\delta \in (0,\epsilon)$ such that $(1-\delta)b(k,d) > (1+\delta)(1-\epsilon)b(k,d)$ and $(1-\epsilon)b(k,d) + (1-\delta)(1-\epsilon)a(k,d) > d(d+1)$. For $\mathcal{H} \geq 1$ large enough, it follows from the definitions of $a(k,d)$ and $b(k,d)$ that

$$\begin{aligned}\sum_{\mathcal{H}'\in B(k,d,\mathcal{H})} |A(k,d,\mathcal{H}')| &\geq \sum_{\substack{\mathcal{H}'\in B(k,d,\mathcal{H})\\ \mathcal{H}'\geq\mathcal{H}^{1-\epsilon}}} |A(k,d,\mathcal{H}')| \\ &\geq \left(\mathcal{H}^{(1-\delta)b(k,d)} - \mathcal{H}^{(1+\delta)(1-\epsilon)b(k,d)}\right)\mathcal{H}^{(1-\delta)(1-\epsilon)a(k,d)}.\end{aligned}$$

As $(1-\delta)b(k,d) > (1+\delta)(1-\epsilon)b(k,d)$ and $\delta < \epsilon$, the right-hand side grows asymptotically faster than

$$\mathcal{H}^{(1-\epsilon)b(k,d)+(1-\delta)(1-\epsilon)a(k,d)}.$$

Since $(1-\epsilon)b(k,d) + (1-\delta)(1-\epsilon)a(k,d) > d(d+1)$, this again contradicts Lemma B.3.1. □

LEMMA B.3.3. *Let $d \in \mathbb{N}$, $\epsilon > 0$, and $k \in \{1,\ldots,d-1\}$. There exists a constant $C = C(k,d,\epsilon)$ such that*

$$|\{\alpha \in A(d-k,d,\mathcal{H});\ \textit{the Galois group of the normal closure of } \mathbb{Q}(\alpha) \textit{ acts}$$
$$\textit{transitively on the } k\textit{-element subsets of the set of conjugates of } \alpha\}| \leq C\mathcal{H}^{\epsilon}$$

for all $\mathcal{H} \geq 1$.

Furthermore, we have

$$|A(k,d,\mathcal{H})| \leq C\mathcal{H}^{\epsilon}$$

for all $\mathcal{H} \geq 1$ with $[\mathbb{Q}(\mathcal{H}^d):\mathbb{Q}] = \binom{d}{k}$.

PROOF. Let $\alpha \in A(d-k,d,\mathcal{H})$ and assume either that the Galois group of the normal closure of $\mathbb{Q}(\alpha)$ acts transitively on the k-element subsets of the set of conjugates of α or that $[\mathbb{Q}(\mathcal{H}^d):\mathbb{Q}] = \binom{d}{d-k}$. Now, for such an α we have $\mathcal{H}^d = H(\alpha)^d = \pm a\alpha_1\cdots\alpha_k$, where $a > 0$ is the leading coefficient of a minimal polynomial of α in $\mathbb{Z}[t]$ and $\alpha_1,\ldots,\alpha_k$ are the conjugates of α that do not lie inside the open unit disk. By assumption, we have $0 < k < d$. We can assume without loss of generality that $\alpha = \alpha_1$ since α_1 determines α up to finitely many possibilities (bounded independently of $\mathcal{H}$).

Now note that

$$a\alpha^k = a\alpha_1^k = \frac{\left(a\prod_{j=1}^k \alpha_j\right)\prod_{i=2}^k\left(a\alpha_{k+1}\left(\prod_{\substack{j=1\\ j\neq i}}^k \alpha_j\right)\right)}{\left(a\prod_{j=2}^{k+1}\alpha_j\right)^{k-1}},$$

where α_{k+1} is a conjugate of α, distinct from the α_j ($j = 1,\ldots,k$) (here we use that $k < d$). The numerator and denominator of the right-hand side are products of conjugates of $\pm\mathcal{H}^d$ by our assumption on either the Galois group of the normal closure of $\mathbb{Q}(\alpha)$ or the degree of $\mathcal{H}^d$. So $a\alpha^k$ is determined by $\mathcal{H}$ up to finitely many possibilities (bounded independently of $\mathcal{H}$), so it can be assumed fixed. The same holds for $a\alpha_j^k$ for all $j = 1,\ldots,d$ by conjugating. And $a\alpha^k$ together with a determines α up to finitely many possibilities (bounded independently of $\mathcal{H}$; here we need that $k > 0$), so it remains to bound the number of possibilities for a.

But $a^{d-k}|b|^k = \prod_{j=1}^d a|\alpha_j|^k$ is already determined up to finitely many possibilities (bounded independently of $\mathcal{H}$), where b is the constant coefficient of a minimal polynomial of α in $\mathbb{Z}[t]$, and a has to divide this natural number as $k < d$. Since

$|\prod_{j=1}^{d} a\alpha_j^k| \leq \mathcal{H}^{d^2}$, it follows from well-known bounds for the divisor function that there are at most $C'(d,\epsilon)\mathcal{H}^{\epsilon}$ possibilities for a. $\square$

LEMMA B.3.4. *Suppose that $\alpha \in \bar{\mathbb{Q}}$ with $[\mathbb{Q}(\alpha) : \mathbb{Q}] = d$. Let $\alpha_1, \ldots, \alpha_d$ be the algebraic conjugates of α and let $a \in \mathbb{Z}$ be the leading coefficient of a minimal polynomial of α in $\mathbb{Z}[t]$. Let S be a subset of $\{1, \ldots, d\}$. Then $a \prod_{s \in S} \alpha_s$ is an algebraic integer.*

PROOF. Let v be a finite valuation of $\mathbb{Q}(\alpha_1, \ldots, \alpha_d)$. We have

$$\left| a \prod_{s \in S} \alpha_s \right|_v \leq |a|_v \prod_{i=1}^{d} \max\{1, |\alpha_i|_v\}.$$

From the Gauss lemma and the definition of a, we deduce that

$$|a|_v \prod_{i=1}^{d} \max\{1, |\alpha_i|_v\} = 1.$$

As v was arbitrary, the lemma follows. $\square$

B.4. The case $k \in \{1, d-1\}$ or d prime

THEOREM B.4.1. *The following hold:*

(i) $a(1,d) = a(d-1,d) = 0$ if $d \geq 2$;
(ii) $b(k,d) = d(d+1)$ ($d \geq 2$, $0 < k < d$);
(iii) $a(k,d) = 0$ (d prime, $0 < k < d$).

(In particular, all these limits exist.)

PROOF. (i) This follows from Lemma B.3.3 as the Galois group of the normal closure of $\mathbb{Q}(\alpha)$ always acts transitively on the 1-element and the $(d-1)$-element subsets of the set of conjugates of α.

(ii) It follows from Lemma B.3.1 that $|B(k,d,\mathcal{H})| = \mathcal{O}\left(\mathcal{H}^{d(d+1)}\right)$ for $\mathcal{H} \geq 1$. If the equality in (ii) were false or the limit $b(k,d)$ did not exist, it would therefore follow that there is some $\epsilon > 0$ such that there exist arbitrarily large $\mathcal{H} \geq 1$ such that $|B(k,d,\mathcal{H})| \leq \mathcal{H}^{d(d+1)-\epsilon}$. It follows from Lemma B.3.3 that the number of $\alpha \in A(k,d,\mathcal{H}')$ with the full symmetric group S_d as Galois group is bounded by $C\mathcal{H}'^{\frac{\epsilon}{2}}$ for some constant C independent of $\mathcal{H}' \in [1,\infty)$. Furthermore, the number of $\alpha \in \bigcup_{\mathcal{H}' \leq \mathcal{H}} A(k,d,\mathcal{H}')$ with Galois group not isomorphic to the full symmetric group is of growth order $o\left(\mathcal{H}^{d(d+1)}\right)$ (see [182]). But by Lemma B.3.1, the number of α of degree d with precisely k conjugates inside the open unit disk and height at most $\mathcal{H}$ grows asymptotically like some constant positive multiple of $\mathcal{H}^{d(d+1)}$, which yields a contradiction for $\mathcal{H}$ large enough.

(iii) We follow a similar strategy as in the proof of Lemma B.3.3. Let d be a prime number, $0 < k < d$, $\mathcal{H} \in [1,\infty)$, and $\alpha \in A(k,d,\mathcal{H})$. The Galois group of the normal closure of $\mathbb{Q}(\alpha)$ must contain an element of order d since d is prime and the Galois group acts transitively on the d-element set of conjugates of α. Since d is prime, an element of order d must act as a d-cycle on the conjugates of α. If these conjugates are $\alpha_1, \ldots, \alpha_d$, we can assume without loss of generality that this d-cycle acts on them by acting on the indices as $(12 \cdots (d-1)d)$. We have $\mathcal{H}^d = H(\alpha)^d = \pm a \prod_{i \in I} \alpha_i$ for some $I \subset \{1, \ldots, d\}$ with $|I| = d-k$ and $a \in \mathbb{N}$ the leading coefficient of a minimal polynomial of α in $\mathbb{Z}[t]$. We aim to write some l-th power of $a\alpha_1^{d-k}$ as a quotient of products of conjugates of $\pm\mathcal{H}^d$, where l and the number of conjugates that appear are bounded in terms of k and d only. Once this is achieved, we can conclude as in the proof of Lemma B.3.3.

To a (formal) product $\prod_{i=1}^d \alpha_i^{e_i}$ with $e_i \in \mathbb{Z}$ we associate a vector $(e_1, \ldots, e_d) \in \mathbb{Z}^d$. Let $v \in \mathbb{Z}^d$ be the vector associated to $\prod_{i\in I} \alpha_i$. Consider the $\mathbb{Z}$-module Λ generated by $A^i v$ ($i = 0, \ldots, d-1$), where A is a permutation matrix corresponding to the cycle $(12\cdots d)$. If finite (which we will later prove it to be), the index $[\mathbb{Z}^d : \Lambda]$ can be bounded by $(d-k)^{\frac{d}{2}}$ through an application of Hadamard's determinant inequality. We could then deduce that $(n, 0, \ldots, 0) \in \Lambda$ for some natural number $n \leq (d-k)^{\frac{d}{2}}$.

We see that $d-k$ must divide n since $d-k$ divides the sum of the coordinates of every element of Λ. Hence we have $n = (d-k)l$ with $l \in \mathbb{N}$ bounded by $(d-k)^{\frac{d}{2}-1}$. The expression of $(n, 0, \ldots, 0)$ as a linear combination of the $A^i v$ is necessarily unique and the coefficients of the $A^i v$ in this linear combination can also be bounded in absolute value in terms of k and d only ($i = 0, \ldots, d-1$). Translating all of this into terms of products of conjugates of $\pm\mathcal{H}^d$ yields that $(a\alpha_1^{d-k})^l$ can be written as a quotient of products of conjugates of $\pm\mathcal{H}^d$ for some natural number $l \leq (d-k)^{\frac{d}{2}-1}$, where the number of conjugates that appears is bounded in terms of k and d only as we wanted.

It remains to prove that $[\mathbb{Z}^d : \Lambda] < \infty$. Equivalently, we can show that the vector subspace V of $\mathbb{C}^d$ generated by the $A^i v$ ($i = 0, \ldots, d-1$) has dimension d. Over $\mathbb{C}$, the matrix A is diagonalizable and we have $\mathbb{C}^d = \bigoplus_{i=0}^{d-1} W_{\zeta^i}$, where ζ is a primitive d-th root of unity and

$$W_\lambda = \{w \in \mathbb{C}^d; Aw = \lambda w\} \quad (\lambda \in \mathbb{C}).$$

The vector subspace V is A-invariant and so $V = \bigoplus_{i=0}^{d-1} (V \cap W_{\zeta^i})$. It cannot be contained in W_1 since $Av \neq v$ (here we use that $0 < k < d$), so there exists some $j \in \{1, \ldots, d-1\}$ with $V \cap W_{\zeta^j} \neq \{0\}$. As $\dim W_{\zeta^i} = 1$ for all i, it follows that $W_{\zeta^j} \subset V$. Since V is defined over $\mathbb{Q}$, it follows by conjugating that $\bigoplus_{i=1}^{d-1} W_{\zeta^i} \subset V$. But $0 \neq \sum_{i=0}^{d-1} A^i v \in V \cap W_1$, so $W_1 \subset V$ as well. It follows that $V = \bigoplus_{i=0}^{d-1} W_{\zeta^i} = \mathbb{C}^d$. $\square$

LEMMA B.4.2. *Let $d \in \mathbb{N}$, $\epsilon > 0$, and $k \in \{1, \ldots, d-1\}$. There exists a constant $C = C(k, d, \epsilon)$ such that*

$$|\{\alpha \in A(k, d, \mathcal{H}); \text{ the Galois group of the normal closure of } \mathbb{Q}(\alpha) \text{ acts } 2\text{-transitively on the conjugates of } \alpha\}| \leq C\mathcal{H}^\epsilon$$

for all $\mathcal{H} \geq 1$.

PROOF. Let $\mathcal{H} \in [1, \infty)$ and $\alpha \in A(k, d, \mathcal{H})$ such that the Galois group of the normal closure of $\mathbb{Q}(\alpha)$ acts 2-transitively on the conjugates of α. Let $\alpha_1, \ldots, \alpha_d$ be the conjugates of α. We want to mimick the proof of Theorem B.4.1(iii). We have $\mathcal{H}^d = \pm a \prod_{i\in I} \alpha_i$ for some $I \subset \{1, \ldots, d\}$ with $|I| = d-k$ and $a \in \mathbb{N}$ the leading coefficient of a minimal polynomial of α in $\mathbb{Z}[t]$. The Galois group of the normal closure of $\mathbb{Q}(\alpha)$ can be identified with a subgroup G of the symmetric group S_d. To a (formal) product $\prod_{i=1}^{d}\alpha_i^{e_i}$ with $e_i\in\mathbb{Z}$ we again associate a vector $(e_1,\ldots,e_d)\in\mathbb{Z}^d$. The group G then acts on $\mathbb{Z}^d$ by permuting the coordinates. We will denote the vector associated to $\prod_{i\in I}\alpha_i$ by v. As we have seen in the proof of Theorem B.4.1(iii), it suffices to show that the vector space V generated over $\mathbb{Q}$ by the gv for $g\in G$ must be $\mathbb{Q}^d$ in order to prove the lemma.

Certainly, this vector space is G-invariant. Since G acts 2-transitively, we know that there are only 4 G-invariant vector subspaces of $\mathbb{Q}^d$, i.e. $\{0\}$, $\mathbb{Q}(1,1,1,\ldots,1)$,

$$\mathbb{Q}(1,-1,0,\ldots,0)\oplus\mathbb{Q}(0,1,-1,0,\ldots,0)\oplus\cdots\oplus\mathbb{Q}(0,\ldots,0,1,-1),$$

and $\mathbb{Q}^d$ (see [166], Exercise 2.6). We can immediately exclude the first two since neither of them contains the vector v. Furthermore, the vector $\sum_{g\in G}gv$ is non-zero and lies in $\mathbb{Q}(1,1,1,\ldots,1)$, so we can also exclude the third one. It follows that $V=\mathbb{Q}^d$ and we are done. $\square$

THEOREM B.4.3. *Let $d\geq 2$. For every $\epsilon>0$, there is $\mathcal{H}_0=\mathcal{H}_0(d,\epsilon)\in\mathbb{R}$ such that*

$$\frac{|\{\alpha\in\mathbb{C};[\mathbb{Q}(\alpha):\mathbb{Q}]=d,H(\alpha)\leq\mathcal{H}\}|}{|\{H(\alpha);\alpha\in\mathbb{C},[\mathbb{Q}(\alpha):\mathbb{Q}]=d,H(\alpha)\leq\mathcal{H}\}|}\leq\mathcal{H}^{\epsilon}$$

for all $\mathcal{H}\geq\mathcal{H}_0$.

Thus, the height function together with the degree is in some sense "almost injective" if the degree is at least 2. The theorem is patently wrong for $d=1$, where the left-hand side grows linearly in $\mathcal{H}$.

PROOF. First, we can replace the numerator in the inequality by the cardinality of the set

$$\{\alpha\in\mathbb{C};[\mathbb{Q}(\alpha):\mathbb{Q}]=d,H(\alpha)\leq\mathcal{H},\ \text{precisely one conjugate of }\alpha\ \text{lies outside the open unit disk}\}.$$

Why? By Lemma B.3.1, the number of α of degree d with precisely one conjugate outside the open unit disk and height at most $\mathcal{H}$ grows asymptotically like some constant positive multiple of $\mathcal{H}^{d(d+1)}$. Because of Lemma B.3.1, applied for all $k\in\{0,\ldots,d\}$ (or thanks to the main result of [98]), demanding that α is in this set then changes the left-hand side by some factor bounded from below by some $c=c(d)>0$ for $\mathcal{H}$ large enough.

Let

$$B(d;\mathcal{H})=\{H(\alpha);\alpha\in\mathbb{C},[\mathbb{Q}(\alpha):\mathbb{Q}]=d,H(\alpha)\leq\mathcal{H}\},$$

then we can rewrite our new numerator as

$$\sum_{\tilde{\mathcal{H}}\in B(d;\mathcal{H})}|\{\alpha\in\mathbb{C};[\mathbb{Q}(\alpha):\mathbb{Q}]=d,H(\alpha)=\tilde{\mathcal{H}},\ \text{precisely one conjugate of }\alpha\ \text{lies outside the open unit disk}\}|.$$

By Theorem B.4.1(i) each summand here is bounded by $C\mathcal{H}^{\frac{\epsilon}{2}}$ for some $C=C(d,\epsilon)$ and we are done. $\square$

B.5. The case $\gcd(k,d)=1$

The following theorem gives a useful unconditional upper bound. We will see later that the exponent in this bound is indeed sharp for every choice of (k,d).

THEOREM B.5.1. *Let $d \in \mathbb{N}$, $\epsilon > 0$, and $k \in \{1,\dots,d-1\}$. There exists C, depending only on d, k, and ϵ, such that for all $\mathcal{H} \geq 1$ we have*

$$|A(k,d,\mathcal{H})| \leq C\mathcal{H}^{d(\gcd(k,d)-1)+\epsilon}.$$

In particular, $a(k,d)=0$ if $\gcd(k,d)=1$.

Let us note at this stage that one might hope a priori to prove that $a(k,d)=0$ for all $d \in \mathbb{N}$ and $k \in \{1,\dots,d-1\}$ with $\gcd(k,d)=1$ by showing the following: For any transitive subgroup G of the symmetric group S_d and any vector $v \in \mathbb{Q}^d$ with exactly k entries equal to 1 and $d-k$ entries equal to 0, the set Gv generates $\mathbb{Q}^d$. Unfortunately, this statement is wrong. One can construct a counterexample with G equal to the subgroup generated by the d-cycle $(12\cdots d)$ from any counterexample to the following statement: Any sum of k distinct d-th roots of unity is non-zero. If we denote $e^{\frac{2\pi\sqrt{-1}}{n}}$ by ζ_n for $n \in \mathbb{N}$, then a construction by Rédei (see [148], Satz 9) yields counterexamples like

$$0 = (-1)+(-1)(-1) = \zeta_2 + \left(\sum_{i=1}^{2}\zeta_3^i\right)\left(\sum_{j=1}^{6}\zeta_7^j\right),$$

where the right-hand side is a sum of 13 distinct 42-nd roots of unity. If G is a 2-transitive subgroup of S_d, then it follows from the proof of Lemma B.4.2 that the statement is correct.

PROOF. Let $\mathcal{H} \geq 1$ and let $\alpha \in A(k,d,\mathcal{H})$. Let $\alpha_1,\dots,\alpha_d$ be the conjugates of α, numbered so that $H(\alpha)^d = a\alpha_1\cdots\alpha_{d-k}$, where a is the (non-zero) leading coefficient of a minimal polynomial of α in $\mathbb{Z}[t]$.

We claim that the coefficients of the polynomial $\prod_{i=1}^{d-k}(t-\alpha_i)$ belong to $\mathbb{Q}(\mathcal{H}^d)$. If not, there would exist an element $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ that fixes $\mathcal{H}^d$, but does not fix the set $\{\alpha_1,\dots,\alpha_{d-k}\}$. But this immediately yields a contradiction since

$$\left|\prod_{s\in S}\alpha_s\right| < |\alpha_1\cdots\alpha_{d-k}|$$

for every subset S of $\{1,\dots,d\}$ of cardinality $d-k$ that is not equal to $\{1,\dots,d-k\}$.

This also implies that the norm of $H(\alpha)^d = \mathcal{H}^d$ with respect to the field extension $\mathbb{Q}(\mathcal{H}^d)/\mathbb{Q}$ is equal to $a^{[\mathbb{Q}(\mathcal{H}^d):\mathbb{Q}]}\prod_{I\in\mathcal{I}}\prod_{\beta\in I}\beta$, where $\mathcal{I}$ is the orbit of $\{\alpha_1,\dots,\alpha_{d-k}\}$ under the Galois group of the normal closure of $\mathbb{Q}(\alpha)$ and the cardinality of $\mathcal{I}$ is $[\mathbb{Q}(\mathcal{H}^d):\mathbb{Q}]$. Since the Galois group acts transitively on $\{\alpha_1,\dots,\alpha_d\}$, the norm of $\mathcal{H}^d$ is equal to

$$a^{[\mathbb{Q}(\mathcal{H}^d):\mathbb{Q}]}(\alpha_1\cdots\alpha_d)^{\left(1-\frac{k}{d}\right)[\mathbb{Q}(\mathcal{H}^d):\mathbb{Q}]}. \tag{B.5.1}$$

In particular, d divides $(d-k)[\mathbb{Q}(\mathcal{H}^d):\mathbb{Q}]$. Since $k>0$ and $a\alpha_1\cdots\alpha_d \in \mathbb{Z}$, we have that a divides the norm of $H(\alpha)^d = \mathcal{H}^d$ in $\mathbb{Z}$. So the number of possibilities for a is bounded by $C_1\mathcal{H}^{\frac{\epsilon}{3}}$ for some constant C_1, depending only on d and ϵ. Hence we can assume that $a \in \mathbb{Z}\backslash\{0\}$ is fixed.

Let F be the normal closure of $\mathbb{Q}(\mathcal{H}^d)$. Set $l = [F(\alpha) : F]$. We will make use of the following simple facts: If K_2/K_1 is a Galois extension of fields of characteristic 0 within a fixed algebraic closure $\overline{K_1}$ and $\xi \in \overline{K_1}$, then $[K_2(\xi) : K_2]$ divides $[K_1(\xi) : K_1]$. Furthermore, if η is a conjugate of ξ over K_1, then $[K_2(\eta) : K_2] = [K_2(\xi) : K_2]$. We deduce that $[F(\alpha) : F] = [F(\alpha_i) : F]$ divides $[\mathbb{Q}(\mathcal{H}^d, \alpha_i) : \mathbb{Q}(\mathcal{H}^d)]$ $(i = 1, \ldots, d-k)$ and divides $[\mathbb{Q}(\alpha) : \mathbb{Q}] = d$. But by the above, $d - k$ is the sum of some of the $[\mathbb{Q}(\mathcal{H}^d, \alpha_i) : \mathbb{Q}(\mathcal{H}^d)]$ (one for each irreducible factor of $\prod_{i=1}^{d-k}(t - \alpha_i)$ in $\mathbb{Q}(\mathcal{H}^d)[t]$). So l divides $d - k$ and d, hence divides $\gcd(k, d)$.

As the number of choices for l is bounded in terms of only k and d, we will from now on assume it fixed. Let $I \subset \{\alpha_1, \ldots, \alpha_d\}$ be the subset of conjugates of α over F (of cardinality l). For $j \in \{1, \ldots, l\}$, we set

$$\gamma_j = a \sum_{J\subset I, |J|=j} \prod_{\beta\in J} \beta.$$

All the γ_j lie in the fixed number field F that is determined uniquely by $\mathcal{H}$ and d. We deduce from Lemma B.3.4 that the γ_j are algebraic integers $(j = 1, \ldots, l)$.

The orbit of I under $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ consists of $\frac{d}{l}$ pairwise disjoint sets $I = I_1, \ldots, I_{\frac{d}{l}}$. We calculate that the $F/\mathbb{Q}$-norm of γ_j is equal to

$$\left(a^{\frac{d}{l}} \prod_{s=1}^{\frac{d}{l}} \sum_{J\subset I_s, |J|=j} \prod_{\beta\in J} \beta\right)^{\frac{[F:\mathbb{Q}]l}{d}}.$$

By Lemma B.3.4, the number

$$N_j = a \prod_{s=1}^{\frac{d}{l}} \sum_{J\subset I_s, |J|=j} \prod_{\beta\in J} \beta$$

is a rational integer and together with a it completely determines the norm of γ_j.

If $j = l$, then $N_j = N_l$ divides the norm of $\mathcal{H}^d$ with respect to the field extension $\mathbb{Q}\left(\mathcal{H}^d\right)/\mathbb{Q}$ by (B.5.1) since $k < d$. Therefore, N_l is already determined up to $C_2\mathcal{H}^{\frac{\epsilon}{3}}$ possibilities, where C_2 depends only on ϵ, d, and k. If $j \in \{1, \ldots, l-1\}$, then N_j is at least bounded in absolute value by $C_3\mathcal{H}^d$, where C_3 depends only on d and k.

Since the algebraic integers γ_j $(j = 1, \ldots, l)$ lie in the given number field F of degree at most $d!$ and their height is bounded by some multiple of $\mathcal{H}^d$ that depends only on d and k, we can argue as in the proof of Proposition 2.5 in [30] to find that the number of possibilities for each of them, if their norm is fixed, is bounded by $C_4\mathcal{H}^{\frac{\epsilon}{3l}}$, where C_4 depends only on d, k, and ϵ. The theorem now follows from $l \leq \gcd(k, d)$ since α is determined up to conjugation by l, a, and the γ_j $(j = 1, \ldots, l)$. $\square$

B.6. The case $\gcd(k, d) > 1$

One might be tempted to conjecture that $a(k, d) = 0$ for all $d \geq 2$ and $0 < k < d$, but this is not true.

THEOREM B.6.1. *The limit* $a(2,4)$ *does not exist. For every* $\kappa \in [0,4]$, *there exists a sequence* $(\mathcal{H}_n)_{n\in\mathbb{N}}$ *in* $B(2,4)$ *such that*

$$\lim_{n\to\infty} \mathcal{H}_n = \infty$$

and

$$\lim_{n\to\infty} \frac{\log|A(2,4,\mathcal{H}_n)|}{\log \mathcal{H}_n} = \kappa.$$

PROOF. Let $\kappa \in [0,4]$. We fix $m \in \mathbb{N}$ prime with $m \not\equiv 1 \mod 4$ and denote its positive square root by $\sqrt{m}$. We define $\overline{u_1 + u_2\sqrt{m}} = u_1 - u_2\sqrt{m}$ $(u_1, u_2 \in \mathbb{Q})$.

If $\kappa < 4$, we apply Bertrand's postulate to find a prime number $b_2 \in \mathbb{N}$ such that

$$m^{\frac{\kappa}{8-2\kappa}} \leq |b_2| \leq 2m^{\frac{\kappa}{8-2\kappa}}. \tag{B.6.1}$$

We then set $\beta = b_1 + b_2\sqrt{m}$, where $b_1 \in \{[b_2\sqrt{m}], [b_2\sqrt{m}]+1\}$ is not divisible by b_2. After maybe replacing β, b_1, b_2 by $-\beta, -b_1, -b_2$, we can assume that $0 < \bar{\beta} < 1$.

If $\kappa = 4$, we take $m = 2$ and $\beta = (3+2\sqrt{2})^r$ for some $r \in \mathbb{N}$. The natural numbers b_1, b_2 are then defined by $\beta = b_1 + b_2\sqrt{2}$. We automatically have that $0 < \bar{\beta} < 1$.

If $\kappa > 0$, we assume that

$$|\beta| \geq 4\sqrt{m} + 8 \tag{B.6.2}$$

by choosing m or r sufficiently large. If $\kappa = 0$, we assume that $m \geq 5$. We set $\mathcal{H} = |\beta|^{\frac{1}{4}}$.

We record that

$$\mathcal{H}^4 = |\beta| \leq 3\sqrt{m}|b_2| \leq 6m^{\frac{1}{2}+\frac{\kappa}{8-2\kappa}} = 6m^{\frac{4}{2(4-\kappa)}} \quad (\kappa < 4) \tag{B.6.3}$$

as well as

$$m^{\frac{4}{2(4-\kappa)}} = m^{\frac{\kappa}{8-2\kappa}+\frac{1}{2}} \leq |b_2|\sqrt{m} \leq |\beta| = \mathcal{H}^4 \quad (\kappa < 4)$$

because of (B.6.1) and hence

$$m \leq \mathcal{H}^{2(4-\kappa)} \quad (\kappa < 4). \tag{B.6.4}$$

Let $\epsilon > 0$. We will prove that there exist positive constants $\mathcal{H}_0, c, \mathcal{H}_1, C$ such that the constants $\mathcal{H}_0$ and c depend only on κ, the constants $\mathcal{H}_1$ and C depend only on κ and ϵ,

$$|A(2,4,\mathcal{H})| \leq C\mathcal{H}^{\kappa+\epsilon}$$

if $\mathcal{H} \geq \mathcal{H}_1$, and

$$|A(2,4,\mathcal{H})| \geq c\mathcal{H}^{\kappa}$$

if $\mathcal{H} \geq \mathcal{H}_0$. The theorem then follows since $\mathcal{H}$ tends to infinity as m or r respectively tend to infinity.

We first prove the upper bound. Let $\alpha \in A(2,4,\mathcal{H})$. Let $\alpha_1, \ldots, \alpha_4$ be the conjugates of α, ordered such that $|\alpha_1|, |\alpha_2| \geq 1$, and let $a > 0$ be the leading coefficient of a minimal polynomial of α in $\mathbb{Z}[t]$. It follows that $\beta = \pm\mathcal{H}^4 \in \{\pm a\alpha_1\alpha_2\}$.

Let F be the fixed field of the stabilizer H of $\{\alpha_1, \alpha_2\}$ in the Galois group G of the normal closure of $\mathbb{Q}(\alpha)$. Arguing as in the proof of Theorem B.5.1, we can show that every $\sigma \in G$ which fixes β must lie in H. It follows that $F = \mathbb{Q}(\beta)$ and $\overline{\beta} \in \{\pm a\alpha_3\alpha_4\}$. We deduce from Lemma B.3.4 that a divides $a^2\prod_{j=1}^4 \alpha_j = \beta\overline{\beta}$ in

$\mathbb{Z}$. Since $|\overline{\beta}| < 1$, it follows from well-known bounds for the divisor function that the number of possibilities for a is bounded by $C_1\mathcal{H}^{\frac{\epsilon}{4}}$ for a certain constant C_1 that depends only on ϵ.

From now on, we assume that a is fixed and count the number of possibilities for α. We have $a\alpha_1^2 - \gamma\alpha_1 \pm \beta = 0$, where $\gamma = a(\alpha_1 + \alpha_2)$. For a given α_1, there are exactly four possible α. From now on, we assume that $\alpha = \alpha_1$. It then suffices to bound the number of possibilities for γ.

Now γ lies in F, so $\gamma \in \mathbb{Q}(\beta)$. Furthermore, we have that $\gamma = a(\alpha_1+\alpha_2) \in \mathbb{Q}(\beta)$ is an algebraic integer by Lemma B.3.4. Since $m \not\equiv 1 \mod 4$, we have $\gamma \in \mathbb{Z}+\mathbb{Z}\sqrt{m}$, so $\gamma = c_1 + c_2\sqrt{m}$ for some $c_1, c_2 \in \mathbb{Z}$. Let $\tilde{a} = \gcd(c_1, c_2, a)$, $\tilde{c}_1 = \tilde{a}^{-1}c_1$, and $\tilde{c}_2 = \tilde{a}^{-1}c_2$. By the usual bound for the divisor function, the number of possibilities for $\tilde{a}$ is bounded from above by $C_2 a^{\frac{\epsilon}{16}} \leq C_2\mathcal{H}^{\frac{\epsilon}{4}}$ with a constant C_2 that depends only on ϵ. In the following, we assume that $\tilde{a}$ is fixed.

As $\overline{\gamma} = a(\alpha_3 + \alpha_4)$, the integer $c_1^2 - mc_2^2 = \gamma\overline{\gamma}$ is divisible by a thanks to Lemma B.3.4. It follows that $\tilde{a}^2 \gcd(a, \tilde{a}^2)^{-1}(\tilde{c}_1^2 - m\tilde{c}_2^2)$ is divisible by $a' = a \gcd(a, \tilde{a}^2)^{-1}$. As $\tilde{a}^2 \gcd(a, \tilde{a}^2)^{-1}$ and a' are coprime, we deduce that $\tilde{c}_1^2 - m\tilde{c}_2^2$ is divisible by a'. By construction, we have that $\gcd(\tilde{c}_1, \tilde{c}_2, a') = 1$. It follows that $\tilde{c}_2 \neq 0$ unless $a' = 1$.

Furthermore, we know that

$$|\tilde{c}_2| = \tilde{a}^{-1}|c_2| \leq \frac{|\gamma| + |\overline{\gamma}|}{2\tilde{a}\sqrt{m}} \leq \frac{2|\beta| + 2a}{2\tilde{a}\sqrt{m}} \leq \frac{2|\beta|}{\tilde{a}\sqrt{m}}$$

since $|\alpha_3|, |\alpha_4| < 1$, $|\alpha_1|, |\alpha_2| \geq 1$, and $a|\alpha_1\alpha_2| = \mathcal{H}^4 = |\beta|$. Thanks to (B.6.3) and (B.6.4), it follows that

$$|\tilde{c}_2| \leq 12\frac{m^{\frac{4}{2(4-\kappa)}}}{\tilde{a}\sqrt{m}} = 12\frac{m^{\frac{\kappa}{2(4-\kappa)}}}{\tilde{a}} \leq 12\frac{\mathcal{H}^\kappa}{\tilde{a}}, \tag{B.6.5}$$

at least if $\kappa < 4$. If $\kappa = 4$, the same follows from $|\beta| = \mathcal{H}^4$ and $\sqrt{2} \leq 12$.

For a given $\tilde{c}_2$, we have to bound the number of $\tilde{c}_1 \in \mathbb{Z}$ such that $|\tilde{c}_1 - \tilde{c}_2\sqrt{m}| = \tilde{a}^{-1}|\overline{\gamma}| < 2a\tilde{a}^{-1}$ and $\tilde{c}_1^2 - m\tilde{c}_2^2 \equiv 0 \mod a'$. Let $\tilde{m} = \gcd(m, a')$, then $\tilde{m}$ is squarefree and must divide $\tilde{c}_1$. Furthermore, $\tilde{m}$ is uniquely determined by m, a, and $\tilde{a}$, so we can assume it fixed. We set $c_1' = \tilde{c}_1\tilde{m}^{-1}$, $m' = m\tilde{m}^{-1}$, and $a'' = a'\tilde{m}^{-1}$. It follows that $\tilde{m}c_1'^2 \equiv m'\tilde{c}_2^2 \mod a''$. By construction, we have $\gcd(m', a'') = 1$. We also have $\gcd(\tilde{c}_2^2, a'') = 1$ since a common prime divisor of a'' and $\tilde{c}_2$ would have to divide a' and therefore $\tilde{c}_1$, but $\gcd(\tilde{c}_1, \tilde{c}_2, a') = 1$. It follows that $\gcd(\tilde{m}c_1'^2, a'') = 1$ as well.

The number of square roots modulo a'' of a number coprime to a'' is bounded by 2^{s+1}, where s is the number of distinct prime factors of a''. The number of c_1' satisfying $|c_1' - \tilde{c}_2\sqrt{m}\tilde{m}^{-1}| = \tilde{m}^{-1}|\tilde{c}_1 - \tilde{c}_2\sqrt{m}| < 2a(\tilde{a}\tilde{m})^{-1}$ that lie in a given congruence class modulo a'' is at most $4\gcd(a, \tilde{a}^2)\tilde{a}^{-1}$ since $\gcd(a, \tilde{a}^2)\tilde{a}^{-1}$ is a natural number and $a'' \gcd(a, \tilde{a}^2)\tilde{a}^{-1} = a(\tilde{a}\tilde{m})^{-1}$. It follows that the number of c_1' for a given $\tilde{c}_2$ is at most $2^{s+3}\gcd(a, \tilde{a}^2)\tilde{a}^{-1}$. If $a'' \geq 3$, we have $s < \frac{7}{5}\frac{\log a''}{\log\log a''}$ by Théorème 11 in [161]. As the function $x \mapsto \frac{\log x}{\log\log x}$ is strictly monotonically increasing for $x \geq 16$, all natural numbers less than 16 have at most 2 distinct prime factors, and $a'' \leq a \leq \mathcal{H}^4$, we have

$$s \leq \max\left\{2, \frac{\frac{28}{5}\log\mathcal{H}}{\log\log\max\{3, \mathcal{H}\}}\right\}.$$

Thanks to (B.6.5), the number of possibilities for the pair $(\tilde{c}_1, \tilde{c}_2)$ is then bounded from above by

$$\left(24\frac{\mathcal{H}^\kappa}{\tilde{a}}\gcd(a,\tilde{a}^2)\tilde{a}^{-1} + \sqrt{a}\right)\cdot 8 \cdot \max\{4, \mathcal{H}^{\frac{28}{5\log\log\max\{3,\mathcal{H}\}}}\}$$
$$\leq (24\mathcal{H}^\kappa + \mathcal{H}^2)\cdot 8\cdot \max\{4, \mathcal{H}^{\frac{28}{5\log\log\max\{3,\mathcal{H}\}}}\},$$

where we have used that $\tilde{c}_2$ can only be 0 if $a' = 1$, in which case $\gcd(a,\tilde{a}^2)\tilde{a}^{-1} = a\tilde{a}^{-1} \leq \sqrt{a}$. If $\kappa \geq 2$, we can estimate $\mathcal{H}^2 \leq \mathcal{H}^\kappa$.

If $\kappa < 2$, we have to study more closely the case that $\tilde{c}_2 = 0$. We use that a is the leading coefficient of a minimal polynomial of α in $\mathbb{Z}[t]$. If $\tilde{c}_2 = 0$ and $\gamma = c_1$, we can therefore conclude that a divides all coefficients of the polynomial

$$(at^2 - a(\alpha_1+\alpha_2)t + a\alpha_1\alpha_2)(at^2 - a(\alpha_3+\alpha_4)t + a\alpha_3\alpha_4) =$$
$$(at^2 - c_1t \pm \beta)(at^2 - c_1 t \pm \bar{\beta}) \in \mathbb{Z}[t].$$

Here, the sign of β is the same as that of $\overline{\beta}$. In particular, a divides $c_1(\beta+\bar{\beta}) = 2b_1c_1$ as well as $\beta\bar{\beta} = b_1^2 - mb_2^2$.

Let $\delta = \gcd(b_1, b_1^2 - mb_2^2) = \gcd(b_1, mb_2^2)$. Since m is prime and $0 < |b_1| \leq |b_2|\sqrt{m} + 1 \leq 2m^{\frac{4}{2(4-\kappa)}} + 1 < m$ by (B.6.1) for $\mathcal{H} \geq \mathcal{H}_1 = \mathcal{H}_1(\kappa)$, we have $\delta = \gcd(b_1, b_2^2)$. But b_2 is prime and does not divide b_1, so $\delta = 1$. Since any common divisor of a and b_1 must also divide δ, it follows that $\gcd(a, b_1) = 1$.

We deduce that c_1 must be divisible by $a\gcd(a,2)^{-1}$. Since $|c_1| = |\overline{\gamma}| < 2a$, there are at most 8 possibilities for c_1.

Putting everything together, the number of possibilities for α is bounded by

$$2\cdot 4\cdot C_1\mathcal{H}^{\frac{\epsilon}{4}}\cdot C_2\mathcal{H}^{\frac{\epsilon}{4}}\cdot 25\mathcal{H}^\kappa\cdot 8\cdot\max\{4,\mathcal{H}^{\frac{28}{5\log\log\max\{3,\mathcal{H}\}}}\} \leq C\mathcal{H}^{\kappa+\epsilon}$$

for $\mathcal{H} \geq \mathcal{H}_1$ with a constant C that depends only on ϵ.

For the lower bound, we first treat the case $\kappa = 0$, so $\beta = \pm(\tilde{b}_1 + 2\sqrt{m})$ with $\tilde{b}_1 \in \{[2\sqrt{m}], [2\sqrt{m}]+1\}$ odd. The degree of $\sqrt{\beta}$ is 4: Otherwise, $\sqrt{\beta}$ would have to be an element $a_1 + a_2\sqrt{m}$ of $\mathbb{Z}[\sqrt{m}]$, which implies that $a_1^2 + a_2^2 m + 2a_1a_2\sqrt{m} = \beta$, so $a_1, a_2 \in \{\pm 1\}$ and $m + 1 \in \{\pm[2\sqrt{m}], \pm([2\sqrt{m}]+1)\}$. This yields a contradiction with $m \geq 5$. Therefore, we have $\sqrt{\beta} \in A(2,4)$. Since $H(\sqrt{\beta}) = \mathcal{H}$, the lower bound holds with $c = 1$.

We now assume that $\kappa > 0$. We choose $\gamma = c_1 + c_2\sqrt{m}$ with

$$c_2 \in \left\{1, \ldots, \left[\frac{|\beta|}{2\sqrt{m}} - \frac{2}{\sqrt{m}}\right]\right\}$$

and $c_1 = [c_2\sqrt{m}] + 1$. It follows that

$$0 < \gamma \leq 2\sqrt{m}c_2 + 1 \leq |\beta| - 3. \tag{B.6.6}$$

We set $\alpha = \frac{\gamma}{2} + \sqrt{\frac{\gamma^2}{4} - \beta}$, so $\alpha^2 - \gamma\alpha + \beta = 0$. It follows that α is an algebraic integer of degree dividing 4. Note that $\alpha \neq 0$ and $\gamma = \frac{\beta + \alpha^2}{\alpha}$ is uniquely defined by α.

We begin by controlling the cases where $[\mathbb{Q}(\alpha) : \mathbb{Q}] < 4$.

If α were a rational integer, then α would be a common divisor of b_1 and b_2. As b_1 and b_2 are coprime by construction, it would follow that $\alpha = \pm 1$ and therefore

$$|\gamma| = \left|\frac{\beta + \alpha^2}{\alpha}\right| = |\beta + 1| \geq |\beta| - 1.$$

This contradicts (B.6.6). So α cannot be a rational integer.

If α is quadratic, we have $\alpha \in \mathbb{Z}[\sqrt{m}]$ (if not, we could apply an automorphism of $\bar{\mathbb{Q}}$ that sends $\sqrt{m}$ to $-\sqrt{m}$, but leaves α unchanged to the defining equation of α and obtain a contradiction). Therefore, $\gamma^2 - 4\beta$ is a square in $\mathbb{Z}[\sqrt{m}]$. It follows that $(\gamma+\delta)(\gamma-\delta) = 4\beta$ for a certain $\delta \in \mathbb{Z}[\sqrt{m}]$. By using an elementary bound for the divisor function, we deduce that the norms of the ideals generated by $\gamma + \delta$ and $\gamma - \delta$ lie in a set of cardinality at most $C_3|\beta|^{\frac{\kappa}{32}}$ for some constant $C_3 = C_3(\kappa)$. Of course, the norms are also at most equal to the norm of 4β in absolute value. The number of ideals of norm N in a quadratic number field is bounded by the number of natural numbers dividing N. It follows that the ideals themselves lie in a set of cardinality at most $C_4|\beta|^{\frac{\kappa}{16}}$ for a constant C_4 that depends only on κ, so we can assume them to be fixed.

This determines $\gamma + \delta$ and $\gamma - \delta$ up to multiplication by a unit of $\mathbb{Z}[\sqrt{m}]$. This unit is of the form ζu^l, where $\zeta = \pm 1$, $l \in \mathbb{Z}$, and u is fixed (depending on m) and satisfies $H(u) > 1$, so $H(u) \geq h_2 = \min\{H(\xi); \xi \in \mathbb{C}, [\mathbb{Q}(\xi) : \mathbb{Q}] \leq 2, H(\xi) > 1\} > 1$. But using the fact that $|\bar{\gamma}| = |[c_2\sqrt{m}] + 1 - c_2\sqrt{m}| < 1$ and $0 < \bar{\beta} < 1$ together with (B.6.6) and elementary properties of the height, we can bound the height of $\gamma \pm \delta$ from above by

$$2H(\gamma)H(\delta) = 2H(\gamma)\sqrt{H(\gamma^2 - 4\beta)} \leq 4\sqrt{2}H(\gamma)^2 H(\beta) = 4\sqrt{2}|\gamma||\beta|^{\frac{1}{2}} \leq 4\sqrt{2}|\beta|^{\frac{3}{2}}.$$

If we write $\eta' = \eta\zeta u^l$, where η and η' are two possible values for $\gamma + \delta$, then it follows that $h_2^{|l|} \leq H(u)^{|l|} = H(u^l) \leq H(\eta)H(\eta') \leq 32|\beta|^3$ and so $|l|$ is bounded from above by $\frac{\log(32) + 3\log|\beta|}{\log h_2}$. Hence there are at most $C_5 \log|\beta|$ possibilities for the unit and hence for $\gamma + \delta$, where C_5 is an absolute constant. Now $\gamma + \delta$ determines $\gamma - \delta$ since $(\gamma+\delta)(\gamma-\delta) = 4\beta$ and β is fixed. And $\gamma + \delta$ together with $\gamma - \delta$ determines γ, so there are at most $C_5 \log|\beta|$ possibilities for γ as well. It follows that α is quadratic for at most $C_6|\beta|^{\frac{\kappa}{8}} = C_6\mathcal{H}^{\frac{\kappa}{2}}$ choices of γ, where $C_6 = C_6(\kappa)$ depends only on κ.

Summarizing, we find that α has degree < 4 for at most $C_6\mathcal{H}^{\frac{\kappa}{2}}$ choices of γ.

If α has degree 4 over $\mathbb{Q}$, its conjugates are $\frac{\gamma}{2} \pm \sqrt{\frac{\gamma^2}{4} - \beta}$ and $\frac{\bar{\gamma}}{2} \pm \sqrt{\frac{\bar{\gamma}^2}{4} - \bar{\beta}}$.

If $\beta > 0$ and $\frac{\gamma^2}{4} < |\beta|$, then

$$\left|\frac{\gamma}{2} \pm \sqrt{\frac{\gamma^2}{4} - \beta}\right| = \sqrt{|\beta|} > 1.$$

If $\beta > 0$ and $\frac{\gamma^2}{4} \geq |\beta|$, we have

$$\left|\frac{\gamma}{2} \pm \sqrt{\frac{\gamma^2}{4} - \beta}\right| \geq \frac{\gamma}{2} - \sqrt{\frac{\gamma^2}{4} - \beta} > 1,$$

since $\gamma < |\beta| + 1$ and $\gamma \geq 2\sqrt{|\beta|} \geq 2$.

If $\beta < 0$, we have

$$\left|\frac{\gamma}{2} \pm \sqrt{\frac{\gamma^2}{4} - \beta}\right| \geq \sqrt{\frac{\gamma^2}{4} + |\beta|} - \frac{\gamma}{2} > 1,$$

since $\gamma < |\beta| - 1$.

Recall that $\bar{\beta} > 0$. If $|\bar{\gamma}| < 2\sqrt{\bar{\beta}}$, then $\sqrt{\frac{\bar{\gamma}^2}{4} - \bar{\beta}}$ is purely imaginary and

$$\left|\frac{\bar{\gamma}}{2} \pm \sqrt{\frac{\bar{\gamma}^2}{4} - \bar{\beta}}\right| = \sqrt{\bar{\beta}} < 1.$$

Otherwise, we have

$$\left|\frac{\bar{\gamma}}{2} \pm \sqrt{\frac{\bar{\gamma}^2}{4} - \bar{\beta}}\right| \leq \left|\frac{\bar{\gamma}}{2}\right| + \sqrt{\frac{\bar{\gamma}^2}{4}} = |\bar{\gamma}| = \left|[c_2\sqrt{m}] + 1 - c_2\sqrt{m}\right| < 1.$$

So in any case, α has two conjugates inside and two conjugates outside the open unit disk. Finally, we can compute

$$H(\alpha)^4 = \left|\left(\frac{\gamma}{2} + \sqrt{\frac{\gamma^2}{4} - \beta}\right)\left(\frac{\gamma}{2} - \sqrt{\frac{\gamma^2}{4} - \beta}\right)\right| = |\beta|,$$

so $H(\alpha) = |\beta|^{\frac{1}{4}} = \mathcal{H}$.

Thanks to (B.6.2), the number of choices for γ can be estimated as

$$\left[\frac{|\beta|}{2\sqrt{m}} - \frac{2}{\sqrt{m}}\right] \geq \frac{|\beta| - 2\sqrt{m} - 4}{2\sqrt{m}} \geq \frac{|\beta|}{4\sqrt{m}} = \frac{\mathcal{H}^4}{4\sqrt{m}}.$$

We can then use (B.6.4) to deduce that the number of choices for γ is equal to at least $\frac{\mathcal{H}^\kappa}{4}$ if $\kappa < 4$. If $\kappa = 4$, we get that the number of choices for γ is equal to at least $\frac{\mathcal{H}^\kappa}{4\sqrt{2}}$. Hence, the lower bound is proven with $c = \frac{1}{8\sqrt{2}}$ and $\mathcal{H}_0 = (8\sqrt{2}C_6)^{\frac{2}{\kappa}}$. □

In fact, the situation is even worse.

THEOREM B.6.2. *Let $k, d \in \mathbb{N}$ such that $0 < k < d$. Then we have*

$$\limsup_{\substack{\mathcal{H} \in B(k,d) \\ \mathcal{H} \to \infty}} \frac{\log |A(k, d, \mathcal{H})|}{\log \mathcal{H}} = d(\gcd(k, d) - 1).$$

PROOF. Let $l = \gcd(k, d)$. The limes superior is at most equal to $d(l-1)$ by Theorem B.5.1. If $l = 1$, this already proves the theorem, so let us assume that $l \geq 2$.

We fix a totally real number field K of degree $\frac{d}{l}$ that is a Galois extension of $\mathbb{Q}$. Such a K can be constructed as a subfield of $\mathbb{Q}\left(\cos\left(\frac{2\pi}{p}\right)\right)$, where p is prime and $p \equiv 1 \mod \frac{2d}{l}$.

Let $\sigma_1, \ldots, \sigma_{\frac{d}{l}}$ be the embeddings of K into $\mathbb{R}$. The set of elements

$$(\log |\sigma_1(u)|, \ldots, \log |\sigma_{\frac{d}{l}-1}(u)|),$$

where u runs over the units of the ring of integers of K, is a lattice in $\mathbb{R}^{\frac{d}{l}-1}$ by Dirichlet's unit theorem. Using elementary multidimensional diophantine approximation, we find that any lattice in $\mathbb{R}^{\frac{d}{l}-1}$ contains a vector $\left(v_1, \ldots, v_{\frac{d}{l}-1}\right)$ such that $v_i < 0$ for $i \leq \frac{k}{l}$, $v_i > 0$ for $i > \frac{k}{l}$, and $\sum_{i=1}^{\frac{d}{l}-1} v_i < 0$. As every algebraic unit has norm 1, it follows that there exists an algebraic unit $u \in K$ such that $|\sigma_i(u)| < 1$ ($i \leq \frac{k}{l}$) and $|\sigma_i(u)| > 1$ ($\frac{k}{l} < i \leq \frac{d}{l}$).

We fix a prime q that does not ramify in K. For $n \in \mathbb{N}$ sufficiently large, we can suppose that the algebraic integer $\beta = qu^n$ satisfies

$$|\sigma_i(\beta)| < \frac{1}{l} \quad \left(i \leq \frac{k}{l}\right)$$

and

$$|\sigma_i(\beta)| > l \quad \left(i > \frac{k}{l}\right).$$

We have $[\mathbb{Q}(\beta) : \mathbb{Q}] = \frac{d}{l}$ since $\frac{k}{l}$ and $\frac{d}{l}$ are coprime.

Set $P(t) = t^l + q\gamma_1 t^{l-1} + \cdots + q\gamma_{l-1}t + \beta$ for algebraic integers $\gamma_1, \ldots, \gamma_{l-1} \in K$. As q is unramified in K, this polynomial is irreducible in $K[t]$ by Eisenstein's criterion for the principal ideal domain obtained by localizing the ring of integers of K at one of the prime ideals lying over q. Let α be a zero of P. The polynomial P is uniquely determined by α, being its monic minimal polynomial over K. Let $\sigma(P) \in K[t]$ denote the polynomial obtained by applying $\sigma \in \mathrm{Gal}(K/\mathbb{Q})$ to the coefficients of P. Since $[\mathbb{Q}(\beta) : \mathbb{Q}] = \frac{d}{l}$, the $\sigma(P)$ ($\sigma \in \mathrm{Gal}(K/\mathbb{Q})$) are all distinct and hence pairwise coprime. It follows that the minimal polynomial of α over $\mathbb{Q}$ is equal to $\prod_{\sigma \in \mathrm{Gal}(K/\mathbb{Q})} \sigma(P)$ and hence $[\mathbb{Q}(\alpha) : \mathbb{Q}] = d = [\mathbb{Q}(\alpha, \beta) : \mathbb{Q}]$. In particular, $\mathbb{Q}(\alpha)$ must contain $\mathbb{Q}(\beta)$.

We now suppose that

$$|\sigma_i(\gamma_j)| \leq \frac{1}{ql} \quad \left(i \leq \frac{k}{l}, j = 1, \ldots, l-1\right) \tag{B.6.7}$$

and

$$|\sigma_i(\gamma_j)| \leq \frac{|\sigma_i(\beta)|}{ql} \quad \left(i > \frac{k}{l}, j = 1, \ldots, l-1\right). \tag{B.6.8}$$

If σ is some embedding of $\mathbb{Q}(\alpha)$ into $\mathbb{C}$ that extends σ_i ($i \leq \frac{k}{l}$), we find that

$$|\sigma(\alpha)|^l \leq \max\{1, |\sigma(\alpha)|\}^{l-1} \left(|\sigma_i(\beta)| + \sum_{j=1}^{l-1} |q\sigma_i(\gamma_j)|\right) < \max\{1, |\sigma(\alpha)|\}^{l-1}.$$

We deduce that $|\sigma(\alpha)| < 1$.

Similarly, if σ is some embedding of $\mathbb{Q}(\alpha)$ into $\mathbb{C}$ that extends σ_i ($i > \frac{k}{l}$), we find that

$$|\sigma_i(\beta)| \leq \max\{1, |\sigma(\alpha)|\}^l \left(1 + \sum_{j=1}^{l-1} |q\sigma_i(\gamma_j)|\right) < |\sigma_i(\beta)| \max\{1, |\sigma(\alpha)|\}^l$$

and we deduce that $|\sigma(\alpha)| > 1$.

Set $\mathcal{H} = H(\beta)^{\frac{1}{l}}$. The above implies that $H(\alpha) = \mathcal{H}$ and furthermore $\alpha \in A(k, d, \mathcal{H})$ (independently of the choice of $\gamma_1, \ldots, \gamma_{l-1}$). Since $H(\beta)$ goes to infinity with $n \to \infty$, we will have shown the theorem if we can show that there are at least $c\mathcal{H}^d$ possibilities for choosing γ_j with (B.6.7) and (B.6.8) (for each $j = 1, \ldots, l-1$), where c is a positive constant depending only on K and q. Now the map $(\sigma_1, \ldots, \sigma_{\frac{d}{l}})$ embeds the ring of algebraic integers of K as a lattice Λ in $\mathbb{R}^{\frac{d}{l}}$. We count points of this lattice that lie inside an aligned box (as defined in [177]) of volume $H(\beta)^{\frac{d}{l}}(ql)^{-\frac{d}{l}} = \mathcal{H}^d(ql)^{-\frac{d}{l}}$.

Scaling Λ by a factor that depends only on K, we obtain a unimodular lattice Λ'. We have $\frac{d}{l} \geq 2$ as $l \leq k < d$. Our desired claim now follows from the fact that Λ' is admissible (as defined in [177]) and the bound (1.4) in [177] (which is deduced from Theorem 1 in [177] if the lattice is admissible). The lattice Λ' is admissible since every non-zero algebraic integer has norm at least 1 in absolute value. $\square$

B.7. The case $(k, d) = (2, 4)$

In this section, we study more closely the case $(k, d) = (2, 4)$. In this case, there are three possibilities for $[\mathbb{Q}(\mathcal{H}^4) : \mathbb{Q}]$, namely 2, 4, or 6. In the last case, we can apply Lemma B.3.3 to obtain that $|A(2, 4, \mathcal{H})|$ grows more slowly than $\mathcal{H}^\epsilon$ for every $\epsilon > 0$. We will show in Theorem B.7.1 that the same holds in the middle case, where $[\mathbb{Q}(\mathcal{H}^4) : \mathbb{Q}] = 4$.

In the first case, it follows from Theorem B.5.1 that we have $|A(2, 4, \mathcal{H})| \leq C(\epsilon)\mathcal{H}^{4+\epsilon}$ for all such $\mathcal{H}$. However, Theorem B.6.1 shows that one cannot always expect this growth and in fact one cannot obtain a uniform growth rate in $\mathcal{H}$ even after partitioning $A(2, 4)$ into finitely many subsets.

Theorem B.7.1. *Let $\epsilon > 0$. There exists a constant $C = C(\epsilon)$ such that $|A(2, 4, \mathcal{H})| \leq C\mathcal{H}^\epsilon$ for all $\mathcal{H} \geq 1$ with $\left[\mathbb{Q}\left(\mathcal{H}^4\right) : \mathbb{Q}\right] = 4$.*

Proof. All unspecified constants in this proof depend only on ϵ.

If $[\mathbb{Q}(\mathcal{H}^4) : \mathbb{Q}] = 4$ and $\alpha \in A(2, 4, \mathcal{H})$, then the normal closure of $\mathbb{Q}(\alpha)$ is either $\mathbb{Q}(\alpha)$ or a number field of degree 8; otherwise, its Galois group would be naturally isomorphic to the symmetric or the alternating group on 4 elements and we would get $[\mathbb{Q}(\mathcal{H}^4) : \mathbb{Q}] = 6$ (recall that a Galois automorphism of $\bar{\mathbb{Q}}$ can only fix $\mathcal{H}^4$ if it fixes the set of conjugates of α that lie outside the open unit disk). We denote the normal closure of $\mathbb{Q}(\mathcal{H}^4)$ by K.

In the first case, K coincides with the normal closure of $\mathbb{Q}(\alpha)$ as both are equal to $\mathbb{Q}(\alpha)$.

In the second case, the Galois group of the normal closure of $\mathbb{Q}(\alpha)$ is isomorphic to the dihedral group D_4 and $\mathbb{Q}(\mathcal{H}^4)$ is a quartic subfield of that normal closure. If K is not equal to the normal closure of $\mathbb{Q}(\alpha)$, then the extension $\mathbb{Q}(\mathcal{H}^4)$ of $\mathbb{Q}$ is Galois. Suppose now that the conjugates of α are the α_i $(i = 1, \ldots, 4)$ and that the Galois group is generated by field automorphisms acting on the conjugates α_i by acting on their indices as the cycle (1234) and the transposition (13). Since $[\mathbb{Q}(\mathcal{H}^4) : \mathbb{Q}] = 4$, we can assume after a cyclic renumbering that $\mathcal{H}^4 = \pm a\alpha_1\alpha_2$, where $a \in \mathbb{N}$ is the leading coefficient of a minimal polynomial of α in $\mathbb{Z}[t]$. The only subfield of the normal closure of $\mathbb{Q}(\alpha)$ of degree 4 that is Galois over $\mathbb{Q}$ corresponds under the Galois correspondence to the cyclic normal subgroup of D_4 generated by (13)(24).

But this element does not fix $\mathcal{H}^4$ since $|\alpha_1\alpha_2| \geq 1 > |\alpha_3\alpha_4|$. So $\mathbb{Q}(\mathcal{H}^4)$ cannot be Galois over $\mathbb{Q}$ and it follows also in this case that K is equal to the normal closure of $\mathbb{Q}(\alpha)$.

In particular, we always have $\alpha \in K$, where K is uniquely determined by $\mathcal{H}$. We also see that the norm of $\mathcal{H}^4$ in K is equal to $a^4(\alpha_1\alpha_2\alpha_3\alpha_4)^2$ or $a^8(\alpha_1\alpha_2\alpha_3\alpha_4)^4$, which is a non-zero integer divisible by a^2. Of course, this determines a up to $c'\mathcal{H}^{\frac{\epsilon}{2}}$ possibilities. Once a is fixed, the norm of the algebraic integer $a\alpha$ in K, which is equal to $a^4\alpha_1\alpha_2\alpha_3\alpha_4$ or $a^8(\alpha_1\alpha_2\alpha_3\alpha_4)^2$, can be determined up to sign. Since the algebraic integer $a\alpha$ lies in a given number field of degree at most 8, we can then argue as in the proof of Proposition 2.5 in [30] to find that the number of possibilities for $a\alpha$ (and thereby for α) is bounded by $c''\mathcal{H}^{\frac{\epsilon}{2}}$. □

We see that the proof of Theorem B.7.1 works in general as soon as the normal closure of $\mathbb{Q}(\mathcal{H}^d)$ contains α. If we restrict ourselves to α such that $\mathbb{Q}(\alpha)$ is Galois over $\mathbb{Q}$, we can for example prove such a result as soon as $[\mathbb{Q}(\mathcal{H}^d) : \mathbb{Q}] = d$.

B.8. Counting polynomials of given Mahler measure

We can also consider polynomials of degree d with integer coefficients of a given Mahler measure instead of algebraic numbers of degree d of a given height. The difference is of course that we also consider reducible polynomials. For a polynomial $A \in \mathbb{Z}[t]$, we denote its Mahler measure by $M(A)$. If α is an algebraic number of degree d, its (multiplicative) height is equal to the d-th (positive real) root of the Mahler measure of any one of its two minimal polynomials in $\mathbb{Z}[t]$. Together with the properties that $M(a) = |a|$ $(a \in \mathbb{Z})$ and $M(AB) = M(A)M(B)$ $(A, B \in \mathbb{Z}[t])$, this characterizes the Mahler measure uniquely.

For given $d \in \mathbb{N}$, $k \in \{0, \dots, d\}$, and $\mathcal{M} \in [1, \infty)$, we define

$$\begin{gathered}\widetilde{A}(k, d, \mathcal{M}) = \{A \in \mathbb{Z}[t]; \deg A = d, M(A) = \mathcal{M}, \text{ and} \\ \text{precisely } k \text{ zeroes of } A \text{ lie inside the open unit disk}\},\end{gathered}$$

$$\widetilde{B}(k, d) = \bigcup_{\mathcal{M} \geq 1} \{M(A); A \in \widetilde{A}(k, d, \mathcal{M})\},$$

and

$$\widetilde{B}(k, d, \mathcal{M}) = \widetilde{B}(k, d) \cap [1, \mathcal{M}].$$

THEOREM B.8.1. *Let $d \in \mathbb{N}$.*
If $k \in \{0, d\}$, then

$$\lim_{\substack{\mathcal{M} \in \widetilde{B}(k,d) \\ \mathcal{M} \to \infty}} \frac{\log |\widetilde{A}(k, d, \mathcal{M})|}{\log \mathcal{M}} = d \tag{B.8.1}$$

and

$$\lim_{\mathcal{M} \to \infty} \frac{\log |\widetilde{B}(k, d, \mathcal{M})|}{\log \mathcal{M}} = 1. \tag{B.8.2}$$

If $k \in \{1, \dots, d-1\}$, we have

$$\liminf_{\substack{\mathcal{M} \in \widetilde{B}(k,d) \\ \mathcal{M} \to \infty}} \frac{\log |\widetilde{A}(k, d, \mathcal{M})|}{\log \mathcal{M}} = 0, \tag{B.8.3}$$

$$\limsup_{\substack{\mathcal{M}\in\widetilde{B}(k,d)\\ \mathcal{M}\to\infty}} \frac{\log|\widetilde{A}(k,d,\mathcal{M})|}{\log\mathcal{M}} = \max\{k, d-k\}, \tag{B.8.4}$$

and

$$\lim_{\mathcal{M}\to\infty} \frac{\log|\widetilde{B}(k,d,\mathcal{M})|}{\log\mathcal{M}} = d+1. \tag{B.8.5}$$

PROOF. Let $d \in \mathbb{N}$, $k \in \{0,\ldots,d\}$, $\mathcal{M} \in \widetilde{B}(k,d)$, and $\epsilon > 0$. All unspecified constants will depend only on d, k, and ϵ.

Let $A \in \widetilde{A}(k,d,\mathcal{M})$. By factoring A into irreducible factors in $\mathbb{Z}[t]$, we see that $\mathcal{M} = a_0\prod_{i=1}^{s} H(\alpha_i)^{d_i}$ for $a_0 \in \mathbb{N}$ and algebraic numbers α_i of degree $[\mathbb{Q}(\alpha_i):\mathbb{Q}] = d_i$ with precisely k_i conjugates inside the open unit disk. Of course, we then have $\sum_{i=1}^{s} d_i = d$ and $\sum_{i=1}^{s} k_i = k$. The number of possibilities for s and the d_i and k_i is bounded in terms of only d and k, so we can assume that s and the d_i and k_i are fixed. Set $F = \mathbb{Q}(\mathcal{M})$. We claim that $H(\alpha_i)^{d_i} \in F$ for all $i = 1,\ldots,s$. If not, there would exist some $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ such that $\sigma(\mathcal{M}) = \mathcal{M}$, but $\sigma\left(H(\alpha_i)^{d_i}\right) \neq H(\alpha_i)^{d_i}$ for some i. But then it follows that

$$\left|\sigma\left(H(\alpha_i)^{d_i}\right)\right| < |H(\alpha_i)^{d_i}|,$$

while

$$\left|\sigma\left(H(\alpha_j)^{d_j}\right)\right| \leq |H(\alpha_j)^{d_j}|$$

for all $j \neq i$, and so $|\mathcal{M}| = |\sigma(\mathcal{M})| < |\mathcal{M}|$, a contradiction.

Now $H(\alpha_i)^{d_i}$ is an algebraic integer by Lemma B.3.4 and its norm (with respect to the field extension $F/\mathbb{Q}$) divides the norm of $\mathcal{M}$. Since $H(\alpha_i)^{d_i}$ furthermore lies in the fixed number field F of degree bounded in terms of only d and k, we can argue as in the proof of Proposition 2.5 in [30] and deduce that $H(\alpha_i)^{d_i}$ is determined up to $C\mathcal{M}^{\epsilon}$ possibilities, so we can assume that $\mathcal{H}_i = H(\alpha_i)^{d_i}$ is fixed. But then α_i is determined up to $C_i\mathcal{H}_i^{\gcd(k_i,d_i)-1+\epsilon}$ possibilities if $0 < k_i < d_i$ and up to $\widetilde{C}_i\mathcal{H}_i^{d_i+\epsilon}$ possibilities if $k_i \in \{0, d_i\}$ by Theorems B.2.1 and B.5.1. Note that $\prod_{i=1}^{s}\mathcal{H}_i \leq \mathcal{M}$.

Since a_0 divides the norm of $\mathcal{M}$ (with respect to the field extension $F/\mathbb{Q}$), it is determined up to $C'\mathcal{M}^{\epsilon}$ possibilities. All in all, the number of possibilities for A (given a fixed s and fixed d_i and k_i) is bounded from above by $\widetilde{C}\mathcal{M}^{(s+2)\epsilon+\max\{e,f\}}$ with

$$e = \max_i\{\gcd(k_i,d_i)-1; 0 < k_i < d_i\}$$

and

$$f = \max_i\{d_i; k_i \in \{0,d_i\}\}.$$

We see that $e \leq \max_i\{k_i - 1\} \leq k-1 < \max\{k,d-k\}$ and $f \leq \max\{k,d-k\}$. It follows that the number of possibilities for A is bounded from above by $\widetilde{C}\mathcal{M}^{\max\{k,d-k\}+(s+2)\epsilon}$. This proves that the limes superior in (B.8.4) is less than or equal to $\max\{k,d-k\}$.

For the inequality in the other direction, we consider $\mathcal{M} \in \mathbb{N}$ such that $\mathcal{M}^{\frac{1}{k}} \in B(k,k)$ and $\mathcal{M}^{\frac{1}{d-k}} \in B(0,d-k)$. By Lemma B.2.2, all sufficiently large $\mathcal{M} \in \mathbb{N}$ satisfy these conditions. We can then apply Theorem B.2.1(ii) to find many products $A(t)(t-1)^{d-k} \in \widetilde{A}(k,d,\mathcal{M})$ with A equal to a minimal polynomial (in $\mathbb{Z}[t]$) of some

$\alpha \in A\left(k, k, \mathcal{M}^{\frac{1}{k}}\right)$ and $A(t)t^k \in \widetilde{A}(k, d, \mathcal{M})$ with A equal to a minimal polynomial (in $\mathbb{Z}[t]$) of some $\alpha \in A\left(0, d-k, \mathcal{M}^{\frac{1}{d-k}}\right)$. Note that α is determined by A up to k or $d-k$ possibilities respectively. This establishes that the limes superior in (B.8.4) is greater than or equal to $\max\{k, d-k\}$. Hence, equality holds in (B.8.4).

If $k \in \{0, d\}$ and $\mathcal{M} \in \widetilde{B}(k, d)$, then $\mathcal{M} \in \mathbb{N}$ automatically. It then follows from the above that

$$\liminf_{\substack{\mathcal{M} \in \widetilde{B}(k,d) \\ \mathcal{M} \to \infty}} \frac{\log |\widetilde{A}(k, d, \mathcal{M})|}{\log \mathcal{M}} \geq d$$

as well as

$$\limsup_{\substack{\mathcal{M} \in \widetilde{B}(k,d) \\ \mathcal{M} \to \infty}} \frac{\log |\widetilde{A}(k, d, \mathcal{M})|}{\log \mathcal{M}} \leq d.$$

We deduce (B.8.1). In that case, we also have $\widetilde{B}(k, d, \mathcal{M}) = \{n \in \mathbb{N}; n \leq \mathcal{M}\}$ (for $\mathcal{M} \in [1, \infty)$) since $M\left(nt^d\right) = M\left(n(t-1)^d\right) = n$ for $n \in \mathbb{N}$, so (B.8.2) follows as well.

Suppose now that $k \in \{1, \ldots, d-1\}$. It follows from Lemma B.3.1 and [182] that we can find $\alpha \in A(k, d)$ of arbitrarily large height such that the Galois group of the normal closure of $\mathbb{Q}(\alpha)$ is isomorphic to the full symmetric group S_d. Any product of k conjugates of such an algebraic number α has degree $\binom{d}{k}$. We can therefore find arbitrarily large $\mathcal{M} = H(\alpha)^d \in \widetilde{B}(k, d)$ such that $[\mathbb{Q}(\mathcal{M}) : \mathbb{Q}] = \binom{d}{k}$.

Let now $\mathcal{M} \in \widetilde{B}(k, d)$ be arbitrary with $[\mathbb{Q}(\mathcal{M}) : \mathbb{Q}] = \binom{d}{k}$ and let $A \in \widetilde{A}(k, d, \mathcal{M})$. Suppose that A decomposes in $\mathbb{Z}[t]$ as the product of a non-zero integer a_0 and s irreducible factors A_i of degree d_i and with k_i zeroes inside the open unit disk respectively ($i = 1, \ldots, s$). We can then bound the degree of $\mathcal{M}$ from above by $\prod_{i=1}^s \binom{d_i}{k_i}$. Using the combinatorial interpretation of the binomial coefficient, one can see that $\binom{d'}{k'}\binom{d''}{k''} < \binom{d'+d''}{k'+k''}$ if $d', d'' \in \mathbb{N}$, $k' \in \{0, \ldots, d'\}$, $k'' \in \{0, \ldots, d''\}$, and $(k', k'') \notin \{(0,0), (d', d'')\}$. If $s > 1$, this implies that $\prod_{i=1}^s \binom{d_i}{k_i}$ is smaller than $\binom{d}{k}$, and we obtain a contradiction.

We deduce that $s = 1$. Therefore, A must be equal to the product of a non-zero integer a_0 and a minimal polynomial (in $\mathbb{Z}[t]$) of some algebraic number α of degree d. We want to bound the number of possibilities for a_0 and α.

Since a_0 divides the norm of $\mathcal{M}$ (with respect to the field extension $\mathbb{Q}(\mathcal{M})/\mathbb{Q}$), the number of possibilities for a_0 is bounded by $C''\mathcal{M}^\epsilon$. Since $[\mathbb{Q}\left(H(\alpha)^d\right) : \mathbb{Q}] = [\mathbb{Q}(\mathcal{M}) : \mathbb{Q}] = \binom{d}{k}$, the number of possibilities for α, given a_0, is bounded by $C'''H(\alpha)^\epsilon \leq C'''\mathcal{M}^\epsilon$ thanks to Lemma B.3.3. We deduce (B.8.3).

We can deduce from Theorem B.4.1(ii) that the limit in (B.8.5) has to be equal to at least $d+1$ (if it exists). For the inequality in the other direction (which will also imply the existence of the limit), we can use that for $\mathcal{M} \in [1, \infty)$, any $\widetilde{\mathcal{M}} \in \widetilde{B}(k, d, \mathcal{M})$ is equal to $M(A)$ for some $A \in \mathbb{Z}[t]$ of degree d, and that the $d+1$ coefficients of this A are all bounded by $2^d M(A) \leq 2^d \mathcal{M}$ in absolute value thanks to Lemma 1.6.7 in [20]. $\square$

B.9. Dynamics of the height function

In this section, we study the dynamics of the restriction of the height function to $\bar{\mathbb{Q}} \cap \mathbb{R}$.

We start by classifying the periodic points. We define inductively $H^0 = \mathrm{id}$ and $H^n = H \circ H^{n-1}$ ($n \in \mathbb{N}$).

THEOREM B.9.1. *If $n \in \mathbb{N}$ and $\alpha \in \bar{\mathbb{Q}}$ are such that $H^n(\alpha) = \alpha$, then $\alpha = a^b$ for some $a \in \mathbb{N}$ and $b \in \mathbb{Q}$, $b > 0$, and $H(\alpha) = \alpha$.*

The proof of this theorem will be essentially achieved by the following lemma:

LEMMA B.9.2. *If $\alpha \in \bar{\mathbb{Q}}$, then $H(\alpha) \geq H(H(\alpha))$ with equality if and only if $H(\alpha) = a^b$ for some $a \in \mathbb{N}$ and $b \in \mathbb{Q}$, $b > 0$.*

PROOF. Let $\beta = H(\alpha)$. We have $\beta^{[\mathbb{Q}(\alpha):\mathbb{Q}]} = a \prod'_{|\gamma| \geq 1} \gamma$, where a is the leading coefficient of a minimal polynomial of α in $\mathbb{Z}[t]$ and the product runs over all zeroes γ of that minimal polynomial that are at least 1 in absolute value. It is now clear that any conjugate of $\beta^{[\mathbb{Q}(\alpha):\mathbb{Q}]}$ that is not equal to $\beta^{[\mathbb{Q}(\alpha):\mathbb{Q}]}$ is less in absolute value than $\beta^{[\mathbb{Q}(\alpha):\mathbb{Q}]}$. Furthermore, $\beta^{[\mathbb{Q}(\alpha):\mathbb{Q}]}$ is an algebraic integer by Lemma B.3.4.

It follows that $H(\beta)^{[\mathbb{Q}(\alpha):\mathbb{Q}]} = H\left(\beta^{[\mathbb{Q}(\alpha):\mathbb{Q}]}\right) < \beta^{[\mathbb{Q}(\alpha):\mathbb{Q}]}$ and hence $H(H(\alpha)) < H(\alpha)$ unless $[\mathbb{Q}\left(\beta^{[\mathbb{Q}(\alpha):\mathbb{Q}]}\right) : \mathbb{Q}] = 1$, in which case $\beta = a^b$ for some $a \in \mathbb{N}$ and $b \in \mathbb{Q}$, $b > 0$. □

PROOF. (of Theorem B.9.1) Suppose that $H^n(\alpha) = \alpha$ for some $n \in \mathbb{N}$ and $\alpha \in \bar{\mathbb{Q}}$. It follows from Lemma B.9.2 that

$$H^n(\alpha) = H^n(H^n(\alpha)) \leq H(H(H^{n-1}(\alpha))) \leq H(H^{n-1}(\alpha)) = H^n(\alpha).$$

We deduce that equality must hold everywhere and hence $\alpha = H^n(\alpha)$ must be of the desired form. □

Next, we study the possibilities for the forward orbit of a given element.

THEOREM B.9.3. *Let $\alpha \in \bar{\mathbb{Q}}$ and define inductively $\alpha_0 = \alpha$, $\alpha_n = H(\alpha_{n-1})$ ($n \in \mathbb{N}$). Then either there exist $N, a \in \mathbb{N}$ and $b \in \mathbb{Q}$, $b > 0$, such that $\alpha_n = a^b$ for all $n \geq N$ or $\lim_{n\to\infty} \alpha_n = 1$.*

PROOF. Let $d = [\mathbb{Q}(\alpha_1) : \mathbb{Q}]$. We will show by induction on n: Either $\alpha_m = a^b$ for some $m, a \in \mathbb{N}$, $m \leq n$, and $b \in \mathbb{Q}$, $b > 0$ (and then of course the same holds for all α_r with $r \geq m$) or $\alpha_n^{d!^{n-1}}$ is the product of at most $(d! - 1)^{n-1}$ conjugates of α_1 up to sign.

The assertion is trivially true for $n = 1$. Suppose now that it has been proven for all $m \leq n$ ($n \in \mathbb{N}$) and suppose further that no α_m is of the form a^b for some $m, a \in \mathbb{N}$, $m \leq n$, and $b \in \mathbb{Q}$, $b > 0$. It follows that $\alpha_n^{d!^{n-1}}$ is a product of at most $(d! - 1)^{n-1}$ conjugates of α_1 up to sign. Since α_1 is an algebraic integer, so is $\alpha_n^{d!^{n-1}}$. Now $\alpha_n^{d!^{n-1}}$ lies in the normal closure of $\mathbb{Q}(\alpha_1)$ and so its degree divides $d!$. It follows that $\alpha_{n+1}^{d!^n} = H\left(\alpha_n^{d!^{n-1}}\right)^{d!}$ is equal to the product of the absolute values of at most $d!$ conjugates of $\alpha_n^{d!^{n-1}}$. Since the non-real conjugates appear in complex conjugate pairs in the product and the real conjugates are equal to their absolute values up to sign, we deduce that $\alpha_{n+1}^{d!^n}$ is equal to the product of at most $d!$ conjugates of $\alpha_n^{d!^{n-1}}$ up to sign.

But if $\alpha_{n+1}^{d!^n}$ is equal (up to sign) to the product of precisely $d!$ conjugates, then it must be a rational integer and so α_{n+1} is of the form a^b for some $a \in \mathbb{N}$ and $b \in \mathbb{Q}$, $b > 0$. Otherwise, it is equal to a product of at most $d! - 1$ conjugates of $\alpha_n^{d!^{n-1}}$ up to sign and therefore equal to a product of at most $(d! - 1)^n$ conjugates of α_1 up to sign.

If no α_n is of the form a^b for some $a \in \mathbb{N}$ and $b \in \mathbb{Q}$, $b > 0$, then it follows directly that

$$1 \leq \alpha_n = |\alpha_n| \leq \alpha_1^{\left(1-\frac{1}{d!}\right)^{n-1}}$$

for all $n \in \mathbb{N}$ since every conjugate of α_1 is less than or equal to α_1 in absolute value. We deduce that $\lim_{n\to\infty} \alpha_n = 1$. $\square$

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Lebenslauf

Ich, Gabriel Andreas Dill, Bürger von Basel BS und Pratteln BL, wurde am 2. August 1993 in Binningen BL geboren. Meine Eltern sind Barbara Gygli Dill und Dr. Ueli Dill, Gymnasiallehrerin und wissenschaftlicher Bibliothekar, wohnhaft in Basel BS.

In Basel BS besuchte ich von 1999 bis 2002 die Primarschule Christoph Merian, von 2002 bis 2005 die Orientierungsschule Gellert und von 2005 bis 2010 das Gymnasium am Münsterplatz, wo ich mit der Matur abschloss (Schwerpunktfach: Griechisch, Grundlagenfach: Latein, Ergänzungsfach: Anwendungen der Mathematik, Maturaarbeit: Apopilosis divi Silvii sive ludus de Silvio Berlusconi – die Verfussballung des göttlichen Silvio oder Satire auf Silvio Berlusconi).

Von 2010 bis 2015 studierte ich an der Universität Basel Mathematik, wo ich mit dem Master of Science (MSc) abschloss (Masterarbeit: Effective Approximation and Diophantine Applications, betreut von Prof. Dr. David W. Masser). Ich besuchte Vorlesungen, Seminare, Proseminare, Übungen, Praktika und Blockkurse bei folgenden Dozentinnen und Dozenten: PD Dr. Annette A'Campo-Neuen (Mathematik), Prof. Dr. Jérémy Blanc (Mathematik), Prof. Dr. Leonhard Burckhardt (Alte Geschichte), Dr. Nakul Chitnis (Mathematik), Prof. Dr. David Cohen (Mathematik), Prof. Dr. Marcus J. Grote (Mathematik), Prof. Dr. Helmut Harbrecht (Mathematik), Prof. Dr. Hans Christoph Im Hof (Mathematik), Prof. Dr. Hanspeter Kraft (Mathematik), Dr. Christoph Luchsinger (Mathematik), Prof. Dr. David W. Masser (Mathematik), Dr. Thorsten Möller (Informatik), Prof. Dr. Heiko Schuldt (Informatik), Prof. Dr. Thomas Vetter (Informatik), Prof. Dr. Evelina Viada (Mathematik), Prof. Dr. Edzard Visser (Gräzistik), Prof. Dr. Rudolf Wachter (Historisch-vergleichende Sprachwissenschaft), PD Dr. Peter Weidemaier (Mathematik). Von 2012 bis 2014 war ich Hilfsassistent mit Lehrverpflichtungen an der Universität Basel.

Die vorliegende Dissertation entstand von 2016 bis 2019 unter der Leitung von Prof. Dr. Philipp Habegger. Während dieser Zeit war ich Assistent mit Lehrverpflichtungen an der Universität Basel. Im Jahr 2018 verbrachte ich einen dreieinhalbmonatigen Auslandsaufenthalt am Institut Fourier in Grenoble unter der fachlichen Leitung von Dr. Gaël Rémond (Directeur de Recherche CNRS).